SOME APPLICATIONS OF MINIMAL $P_\gamma$-OPEN SETS

ALIAS B. KHALAF AND HARIWAN Z. IBRAHIM

Abstract. We characterize minimal $P_\gamma$-open sets in topological spaces. We show that any nonempty subset of a minimal $P_\gamma$-open set is pre $P_\gamma$-open. As an application of a theory of minimal $P_\gamma$-open sets, we obtain a sufficient condition for a $P_\gamma$-locally finite space to be a pre $P_\gamma$-Hausdorff space.

1. Introduction

Mashhour et al [3], introduced and investigated the notions of preopen sets, and Kasahara [4], defined the concept of an operation on topological spaces. Ogata [5], introduced the concept of $\gamma$-open sets and investigated the related topological properties of the associated topology $\tau_\gamma$ and $\tau$, where $\tau_\gamma$ is the collection of all $\gamma$-open sets.

In this paper, we study fundamental properties of minimal $P_\gamma$-open sets and apply them to obtain some results in topological spaces. Also we give some characterizations of minimal $P_\gamma$-open sets. Moreover, we define and study $P_\gamma$-locally finite spaces and we apply minimal $P_\gamma$-open sets to define pre $P_\gamma$-open sets.

Finally, we show that any $P_\gamma$-locally finite space containing a minimal $P_\gamma$-open subset is pre $P_\gamma$-Hausdorff.

2. Preliminaries

Definition 2.1. [3] A subset $A$ of a topological space $(X, \tau)$ is said to be preopen if $A \subseteq \text{Int}(\text{Cl}(A))$. The family of all preopen sets is denoted by $PO(X, \tau)$.

Definition 2.2. [4] Let $(X, \tau)$ be a topological space. An operation $\gamma$ on the topology $\tau$ is a mapping from $\tau$ to power set $P(X)$ of $X$ such that $V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of $\gamma$ at $V$. It is denoted by $\gamma : \tau \to P(X)$.

Definition 2.3. [5] A subset $A$ of a topological space $(X, \tau)$ is called $\gamma$-open set if for each $x \in A$ there exists an open set $U$ such that $x \in U$ and $\gamma(U) \subseteq A$.

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Definition 2.4. [2] Let $X$ be a space and $A \subseteq X$ a $\gamma$-open set. Then $A$ is called a minimal $\gamma$-open set if $\phi$ and $A$ are the only $\gamma$-open subsets of $A$.

The following definitions and results are obtained from [1]. we defined $\gamma$ to be a mapping on $PO(X)$ into $P(X)$ and $\gamma : PO(X) \rightarrow P(X)$ is called an operation on $PO(X)$, such that $V \subseteq \gamma(V)$ for each $V \in PO(X)$ [1].

Definition 2.5. A subset $A$ of a space $X$ is called $P_\gamma$-open if for each $x \in A$, there exists a preopen set $U$ such that $x \in U$ and $\gamma(U) \subseteq A$.

Definition 2.6. Let $A$ be a subset of $(X, \tau)$, and $\gamma : PO(X) \rightarrow P(X)$ be an operation. Then the $P_\gamma$-closure (resp., $P_\gamma$-interior) of $A$ is denoted by $p_\gamma Cl(A)$ (resp., $p_\gamma Int(A)$) and defined as follows:

1. $p_\gamma Cl(A) = \bigcap\{F : F$ is $P_\gamma$-closed and $A \subseteq F\}$.
2. $p_\gamma Int(A) = \bigcup\{U : U$ is $P_\gamma$-open and $U \subseteq A\}$.

Theorem 2.7. For a point $x \in X$, $x \in p_\gamma Cl(A)$ if and only if for every $P_\gamma$-open set $V$ of $X$ containing $x$, $A \cap V \neq \phi$.

Definition 2.8. An operation $\gamma$ on $PO(X)$ is said to be pre regular if for every preopen sets $U$ and $V$ of each $x \in X$, there exists a preopen set $W$ of $x$ such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Proposition 2.9. Let $\gamma$ be a pre regular operation on $PO(X)$. If $A$ and $B$ are $P_\gamma$-open sets in $X$, then $A \cap B$ is also a $P_\gamma$-open set.

3. Minimal $P_\gamma$-open sets

In view of the definition of minimal $\gamma$-open sets [2], we define minimal $P_\gamma$-open sets as:

Definition 3.1. Let $X$ be a space and $A \subseteq X$ a $P_\gamma$-open set. Then $A$ is called a minimal $P_\gamma$-open set if $\phi$ and $A$ are the only $P_\gamma$-open subsets of $A$.

The following examples show that minimal $P_\gamma$-open sets and minimal $\gamma$-open sets are independent of each other.

Example 3.2. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X\}$. Define an operation $\gamma : PO(X) \rightarrow P(X)$ by $\gamma(A) = A$. The $P_\gamma$-open sets are $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and $X$. Here $\{a\}$ is minimal $P_\gamma$-open but not minimal $\gamma$-open. Also we consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a, b\}, X\}$. Define $\gamma : PO(X) \rightarrow P(X)$ as $\gamma(A) = A$, the set $\{a, b\}$ is minimal $\gamma$-open but not minimal $P_\gamma$-open.

Proposition 3.3. Let $X$ be a space. Then:

1. Let $A$ be a minimal $P_\gamma$-open set and $B$ a $P_\gamma$-open set. Then $A \cap B = \phi$ or $A \subseteq B$, where $\gamma$ is pre regular.
2. Let $B$ and $C$ be minimal $P_\gamma$-open sets. Then $B \cap C = \phi$ or $B = C$, where $\gamma$ is pre regular.
Proof. (1) Let $B$ be a $P_{\gamma}$-open set such that $A \cap B \neq \phi$. Since $A$ is a minimal $P_{\gamma}$-open set and $A \cap B \subseteq A$, we have $A \cap B = A$. Therefore $A \subseteq B$.

(2) If $B \cap C \neq \phi$, then we see that $B \subseteq C$ and $C \subseteq B$ by (1). Therefore $B = C$. \hfill \Box

**Proposition 3.4.** Let $A$ be a minimal $P_{\gamma}$-open set. If $x$ is an element of $A$, then $A \subseteq B$ for any $P_{\gamma}$-open neighborhood $B$ of $x$, where $\gamma$ is pre regular.

**Proof.** Let $B$ be a $P_{\gamma}$-open neighborhood of $x$ such that $A \not\subseteq B$. Since $\gamma$ is pre regular operation, then $A \cap B$ is $P_{\gamma}$-open set such that $A \cap B \subseteq A$ and $A \cap B \neq \phi$. This contradicts our assumption that $A$ is a minimal $P_{\gamma}$-open set.

The following example shows that the condition that $\gamma$ is pre regular is necessary for the above proposition.

**Example 3.5.** Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Define an operation $\gamma$ on $PO(X)$ by

$$\gamma(A) = \begin{cases} A & \text{if } b \in A \\ Cl(A) & \text{if } b \notin A \end{cases}$$

Then calculations show that the operation $\gamma$ is not pre regular. Clearly $A = \{a, c\}$ is a minimal $P_{\gamma}$-open set. Thus for $a \in A$, there is no $P_{\gamma}$-open set $B$ containing $a$ such that $A \subseteq B$.

**Proposition 3.6.** Let $A$ be a minimal $P_{\gamma}$-open set. Then for any element $x$ of $A$, $A = \cap\{B : B$ is $P_{\gamma}$-open neighborhood of $x\}$, where $\gamma$ is pre regular.

**Proof.** By Proposition 3.4 and the fact that $A$ is $P_{\gamma}$-open neighborhood of $x$, we have $A \subseteq \cap\{B : B$ is $P_{\gamma}$-open neighborhood of $x\} \subseteq A$. Therefore we have the result. \hfill \Box

**Proposition 3.7.** Let $A$ be a minimal $P_{\gamma}$-open set in $X$ and $x \in X$ such that $x \notin A$. Then for any $P_{\gamma}$-open neighborhood $C$ of $x$, $C \cap A = \phi$ or $A \subseteq C$, where $\gamma$ is pre regular.

**Proof.** Since $C$ is a $P_{\gamma}$-open set, we have the result by Proposition 3.3. \hfill \Box

**Corollary 3.8.** Let $A$ be a minimal $P_{\gamma}$-open set in $X$ and $x \in X$ such that $x \notin A$. Define $A_x = \cap\{B : B$ is $P_{\gamma}$-open neighborhood of $x\}$. Then $A_x \cap A = \phi$ or $A \subseteq A_x$, where $\gamma$ is pre regular.

**Proof.** If $A \subseteq B$ for any $P_{\gamma}$-open neighborhood $B$ of $x$, then $A \subseteq \cap\{B : B$ is $P_{\gamma}$-open neighborhood of $x\}$. Therefore $A \subseteq A_x$. Otherwise there exists a $P_{\gamma}$-open neighborhood $B$ of $x$ such that $B \cap A = \phi$. Then we have $A_x \cap A = \phi$. \hfill \Box

**Corollary 3.9.** If $A$ is a nonempty minimal $P_{\gamma}$-open set of $X$, then for a nonempty subset $C$ of $A$, $A \subseteq p_{\gamma}Cl(C)$, where $\gamma$ is pre regular.
Proof. Let $C$ be any nonempty subset of $A$. Let $y \in A$ and $B$ be any $P_\gamma$-open neighborhood of $y$. By Proposition 3.4, we have $A \subseteq B$ and $C = A \cap C \subseteq B \cap C$. Thus we have $B \cap C \neq \phi$ and hence $y \in p_\gamma Cl(C)$. This implies that $A \subseteq p_\gamma Cl(C)$. This completes the proof. \qed

Proposition 3.10. Let $A$ be a nonempty $P_\gamma$-open subset of a space $X$. If $A \subseteq p_\gamma Cl(C)$, then $p_\gamma Cl(A) = p_\gamma Cl(C)$, for any nonempty subset $C$ of $A$.

Proof. For any nonempty subset $C$ of $A$, we have $p_\gamma Cl(C) \subseteq p_\gamma Cl(A)$. On the other hand, by supposition we see $p_\gamma Cl(A) \subseteq p_\gamma Cl(p_\gamma Cl(C)) = p_\gamma Cl(C)$ implies $p_\gamma Cl(A) \subseteq p_\gamma Cl(C)$. Therefore we have $p_\gamma Cl(A) = p_\gamma Cl(C)$ for any nonempty subset $C$ of $A$. \qed

Proposition 3.11. Let $A$ be a nonempty $P_\gamma$-open subset of a space $X$. If $p_\gamma Cl(A) = p_\gamma Cl(C)$, for any nonempty subset $C$ of $A$, then $A$ is a minimal $P_\gamma$-open set.

Proof. Suppose that $A$ is not a minimal $P_\gamma$-open set. Then there exists a nonempty $P_\gamma$-open set $B$ such that $B \subseteq A$ and hence there exists an element $x \in A$ such that $x \notin B$. Then we have $p_\gamma Cl(\{x\}) \subseteq (X \setminus B)$ implies that $p_\gamma Cl(\{x\}) \neq p_\gamma Cl(A)$. This contradiction proves the proposition. \qed

Combining Corollary 3.9 and Propositions 3.10 and 3.11, we have:

Theorem 3.12. Let $A$ be a nonempty $P_\gamma$-open subset of space $X$. Then the following are equivalent:

1. $A$ is minimal $P_\gamma$-open set, where $\gamma$ is pre regular.
2. For any nonempty subset $C$ of $A$, $A \subseteq p_\gamma Cl(C)$.
3. For any nonempty subset $C$ of $A$, $p_\gamma Cl(A) = p_\gamma Cl(C)$.

Definition 3.13. A subset $A$ of a space $X$ is called a pre $P_\gamma$-open set if $A \subseteq p_\gamma Int(p_\gamma Cl(A))$. The family of all pre $P_\gamma$-open sets of $X$ will be denoted by $PO_\gamma(X)$.

Definition 3.14. A space $X$ is called pre $P_\gamma$-Hausdorff if for each $x, y \in X$, $x \neq y$ there exist subsets $U$ and $V$ of $PO_\gamma(X)$ such that $x \in U$, $y \in V$, and $U \cap V = \phi$.

Theorem 3.15. Let $A$ be a minimal $P_\gamma$-open set. Then any nonempty subset $C$ of $A$ is a pre $P_\gamma$-open set, where $\gamma$ is pre regular.

Proof. By Corollary 3.9, we have $A \subseteq p_\gamma Cl(C)$ implies $p_\gamma Int(A) \subseteq p_\gamma Int(p_\gamma Cl(C))$. Since $A$ is a $P_\gamma$-open set, we have $C \subseteq A = p_\gamma Int(A) \subseteq p_\gamma Int(p_\gamma Cl(C))$ or $C \subseteq p_\gamma Int(p_\gamma Cl(C))$, that is $C$ pre $P_\gamma$-open. Hence the proof. \qed

Theorem 3.16. Let $A$ be a minimal $P_\gamma$-open set and $B$ be a nonempty subset of $X$. If there exists a $P_\gamma$-open set $C$ containing $B$ such that $C \subseteq p_\gamma Cl(B \cup A)$, then $B \cup D$ is a pre $P_\gamma$-open set for any nonempty subset $D$ of $A$, where $\gamma$ is pre regular.
Proof. By Theorem 3.12 (3), we have \( p_\gamma \text{Cl}(B \cup D) = p_\gamma \text{Cl}(B) \cup p_\gamma \text{Cl}(D) = p_\gamma \text{Cl}(B) \cup p_\gamma \text{Cl}(A) = p_\gamma \text{Cl}(B \cup A) \). By supposition \( C \subseteq p_\gamma \text{Cl}(B \cup A) = p_\gamma \text{Cl}(B \cup D) \) implies \( p_\gamma \text{Int}(C) \subseteq p_\gamma \text{Int}(p_\gamma \text{Cl}(B \cup D)) \). Since \( C \) is a \( P_\gamma \)-open neighborhood of \( B \), namely \( C \) is a \( P_\gamma \)-open such that \( B \subseteq C \), we have \( B \subseteq C = p_\gamma \text{Int}(C) \subseteq p_\gamma \text{Int}(p_\gamma \text{Cl}(B \cup D)) \). Moreover we have \( p_\gamma \text{Int}(A) \subseteq p_\gamma \text{Int}(p_\gamma \text{Cl}(B \cup A)) \), for \( p_\gamma \text{Int}(A) = A \subseteq p_\gamma \text{Cl}(A) \subseteq p_\gamma \text{Cl}(B) \cup p_\gamma \text{Cl}(A) = p_\gamma \text{Cl}(B \cup A) \). Since \( A \) is a \( P_\gamma \)-open set, we have \( D \subseteq A = p_\gamma \text{Int}(A) \subseteq p_\gamma \text{Int}(p_\gamma \text{Cl}(B \cup A)) = p_\gamma \text{Int}(p_\gamma \text{Cl}(B \cup D)) \). Therefore \( B \cup D \subseteq p_\gamma \text{Int}(p_\gamma \text{Cl}(B \cup D)) \) implies \( B \cup D \) is a pre \( P_\gamma \)-open set. \( \square \)

**Corollary 3.17.** Let \( A \) be a minimal \( P_\gamma \)-open set and \( B \) a nonempty subset of \( X \). If there exists a \( P_\gamma \)-open set \( C \) containing \( B \) such that \( C \subseteq p_\gamma \text{Cl}(A) \), then \( B \cup D \) is a pre \( P_\gamma \)-open set for any nonempty subset \( D \) of \( A \), where \( \gamma \) is pre regular.

**Proof.** By assumption, we have \( C \subseteq p_\gamma \text{Cl}(B) \cup p_\gamma \text{Cl}(A) = p_\gamma \text{Cl}(B \cup A) \). By Theorem 3.16, we see that \( B \cup D \) is a pre \( P_\gamma \)-open set. \( \square \)

### 4. Finite \( P_\gamma \)-open sets

In this section, we study some properties of minimal \( P_\gamma \)-open sets in finite \( P_\gamma \)-open sets and \( P_\gamma \)-locally finite spaces.

**Proposition 4.1.** Let \( X \) be a space and \( \emptyset \neq B \) a finite \( P_\gamma \)-open set in \( X \). Then there exists at least one (finite) minimal \( P_\gamma \)-open set \( A \) such that \( A \subseteq B \).

**Proof.** Suppose that \( B \) is a finite \( P_\gamma \)-open set in \( X \). Then we have the following two possibilities:

1. \( B \) is a minimal \( P_\gamma \)-open set.
2. \( B \) is not a minimal \( P_\gamma \)-open set.

In case (1), if we choose \( B = A \), then the proposition is proved.

If the case (2) is true, then there exists a nonempty (finite) \( P_\gamma \)-open set \( B_1 \) which is properly contained in \( B \). If \( B_1 \) is minimal \( P_\gamma \)-open, we take \( A = B_1 \). If \( B_1 \) is not a minimal \( P_\gamma \)-open set, then there exists a nonempty (finite) \( P_\gamma \)-open set \( B_2 \) such that \( B_2 \subseteq B_1 \subseteq B \). We continue this process and have a sequence of \( P_\gamma \)-open sets \( B \) such that \( B_2 \subseteq ... \subseteq B_m \subseteq ... \subseteq B_2 \subseteq B_1 \subseteq B \). Since \( B \) is a finite, this process will end in a finite number of steps. That is, for some natural number \( k \), we have a minimal \( P_\gamma \)-open set \( B_k \) such that \( B_k = A \). This completes the proof. \( \square \)

**Definition 4.2.** A space \( X \) is said to be a \( P_\gamma \)-locally finite space, if for each \( x \in X \) there exists a finite \( P_\gamma \)-open set \( A \) in \( X \) such that \( x \in A \).

**Corollary 4.3.** Let \( X \) be a \( P_\gamma \)-locally finite space and \( B \) a nonempty \( P_\gamma \)-open set. Then there exists at least one (finite) minimal \( P_\gamma \)-open set \( A \) such that \( A \subseteq B \), where \( \gamma \) is pre regular.
Proof. Since $B$ is a nonempty set, there exists an element $x$ of $B$. Since $X$ is a $P_\gamma$-locally finite space, we have a finite $P_\gamma$-open set $B_x$ such that $x \in B_x$. Since $B \cap B_x$ is a finite $P_\gamma$-open set, we get a minimal $P_\gamma$-open set $A$ such that $A \subseteq B \cap B_x \subseteq B$ by Proposition 4.1.

Proposition 4.4. Let $X$ be a space and for any $\alpha \in I$, $B_\alpha$ a $P_\gamma$-open set and $\phi \neq A$ a finite $P_\gamma$-open set. Then $A \cap (\bigcap_{\alpha \in I} B_\alpha)$ is a finite $P_\gamma$-open set, where $\gamma$ is pre regular.

Proof. We see that there exists an integer $n$ such that $A \cap (\bigcap_{\alpha \in I} B_\alpha) = A \cap (\bigcap_{i=1}^{n} B_\alpha)$ and hence we have the result.

Using Proposition 4.4, we can prove the following:

Theorem 4.5. Let $X$ be a space and for any $\alpha \in I$, $B_\alpha$ a $P_\gamma$-open set and for any $\beta \in J$, $A_\beta$ a nonempty finite $P_\gamma$-open set. Then $(\bigcup_{\beta \in J} A_\beta) \cap (\bigcap_{\alpha \in I} B_\alpha)$ is a $P_\gamma$-open set, where $\gamma$ is pre regular.

5. Applications

Let $A$ be a nonempty finite $P_\gamma$-open set. It is clear, by Proposition 3.3 and Proposition 4.1, that if $\gamma$ is pre regular, then there exists a natural number $m$ such that $\{A_1, A_2, ..., A_m\}$ is the class of all minimal $P_\gamma$-open sets in $A$ satisfying the following two conditions:

1. For any $l, n$ with $1 \leq l, n \leq m$ and $l \neq n$, $A_l \cap A_n = \phi$.
2. If $C$ is a minimal $P_\gamma$-open set in $A$, then there exists $l$ with $1 \leq l \leq m$ such that $C = A_l$.

Theorem 5.1. Let $X$ be a space and $\phi \neq A$ a finite $P_\gamma$-open set such that $A$ is not a minimal $P_\gamma$-open set. Let $\{A_1, A_2, ..., A_m\}$ be a class of all minimal $P_\gamma$-open sets in $A$ and $y \in A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$. Define $A_y = \bigcap \{B : B$ is a $P_\gamma$-open neighborhood of $y]\}$. Then there exists a natural number $k \in \{1, 2, ..., m\}$ such that $A_k$ is contained in $A_y$, where $\gamma$ is pre regular.

Proof. Suppose on the contrary that for any natural number $k \in \{1, 2, ..., m\}$, $A_k$ is not contained in $A_y$. By Corollary 3.8, for any minimal $P_\gamma$-open set $A_k$ in $A$, $A_k \cap A_y = \phi$. By Proposition 4.4, $\phi \neq A_y$ is a finite $P_\gamma$-open set. Therefore by Proposition 4.1, there exists a minimal $P_\gamma$-open set $C$ such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, we have $C$ is a minimal $P_\gamma$-open set in $A$. By supposition, for any minimal $P_\gamma$-open set $A_k$, we have $A_k \cap C \subseteq A_k \cap A_y = \phi$. Therefore for any natural number $k \in \{1, 2, ..., m\}$, $C \neq A_k$. This contradicts our assumption. Hence the proof.

Proposition 5.2. Let $X$ be a space and $\phi \neq A$ be a finite $P_\gamma$-open set which is not a minimal $P_\gamma$-open set. Let $\{A_1, A_2, ..., A_m\}$ be a class of all minimal $P_\gamma$-open sets in $A$ and $y \in A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, ..., m\}$ such that for any $P_\gamma$-open neighborhood $B_y$ of $y$, $A_k$ is contained in $B_y$, where $\gamma$ is pre regular.
Proof. This follows from Theorem 5.1, as \( \cap \{ B : B \) is a \( P_\gamma \)-open of \( y \} \subseteq B_y \). Hence the proof.

**Theorem 5.3.** Let \( X \) be a space and \( \phi \neq A \) be a finite \( P_\gamma \)-open set which is not a minimal \( P_\gamma \)-open set. Let \( \{ A_1, A_2, \ldots, A_m \} \) be the class of all minimal \( P_\gamma \)-open sets in \( A \) and \( y \in A \setminus (A_1 \cup A_2 \cup \ldots \cup A_m) \). Then there exists a natural number \( k \in \{ 1, 2, \ldots, m \} \) such that \( y \in p_\gamma Cl(A_k) \), where \( \gamma \) is pre regular.

Proof. It follows from Proposition 5.2, that there exists a natural number \( k \in \{ 1, 2, \ldots, m \} \) such that \( A_k \subseteq B \) for any \( P_\gamma \)-open neighborhood \( B \) of \( y \). Therefore \( \phi \neq A_k \cap A_k \subseteq A_k \cap B \) implies \( y \in p_\gamma Cl(A_k) \). This completes the proof.

**Proposition 5.4.** Let \( \phi \neq A \) be a finite \( P_\gamma \)-open set in a space \( X \) and for each \( k \in \{ 1, 2, \ldots, m \} \), \( A_k \) is a minimal \( P_\gamma \)-open set in \( A \). If the class \( \{ A_1, A_2, \ldots, A_m \} \) contains all minimal \( P_\gamma \)-open sets in \( A \), then for any \( \phi \neq B_k \subseteq A_k \), \( A \subseteq p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \), where \( \gamma \) is pre regular.

Proof. If \( A \) is a minimal \( P_\gamma \)-open set, then this is the result of Theorem 3.12 (2). Otherwise \( A \) is not a minimal \( P_\gamma \)-open set. If \( x \) is any element of \( A \setminus (A_1 \cup A_2 \cup \ldots \cup A_m) \), we have \( x \in p_\gamma Cl(A_1) \cup p_\gamma Cl(A_2) \cup \ldots \cup p_\gamma Cl(A_m) \) by Theorem 5.3. Therefore \( A \subseteq p_\gamma Cl(A_1) \cup p_\gamma Cl(A_2) \cup \ldots \cup p_\gamma Cl(A_m) = p_\gamma Cl(B_1) \cup p_\gamma Cl(B_2) \cup \ldots \cup p_\gamma Cl(B_m) = p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \) by Theorem 3.12 (3).

**Proposition 5.5.** Let \( \phi \neq A \) be a finite \( P_\gamma \)-open set and \( A_k \) is a minimal \( P_\gamma \)-open set in \( A \), for each \( k \in \{ 1, 2, \ldots, m \} \). If for any \( \phi \neq B_k \subseteq A_k \), \( A \subseteq p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \) then \( p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \).

Proof. For any \( \phi \neq B_k \subseteq A_k \) with \( k \in \{ 1, 2, \ldots, m \} \), we have \( p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \subseteq p_\gamma Cl(A) \). Also, we have \( p_\gamma Cl(A) \subseteq p_\gamma Cl(p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m)) = p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \). Therefore we have \( p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \) for any nonempty subset \( B_k \) of \( A_k \) with \( k \in \{ 1, 2, \ldots, m \} \).

**Proposition 5.6.** Let \( \phi \neq A \) be a finite \( P_\gamma \)-open set and for each \( k \in \{ 1, 2, \ldots, m \} \), \( A_k \) is a minimal \( P_\gamma \)-open set in \( A \). If for any \( \phi \neq B_k \subseteq A_k \), \( p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \), then the class \( \{ A_1, A_2, \ldots, A_m \} \) contains all minimal \( P_\gamma \)-open sets in \( A \).

Proof. Suppose that \( C \) is a minimal \( P_\gamma \)-open set in \( A \) and \( C \neq A_k \) for \( k \in \{ 1, 2, \ldots, m \} \). Then we have \( C \cap p_\gamma Cl(A_k) = \phi \) for each \( k \in \{ 1, 2, \ldots, m \} \). It follows that any element of \( C \) is not contained in \( p_\gamma Cl(A_1 \cup A_2 \cup \ldots \cup A_m) \). This is a contradiction to the fact that \( C \subseteq A \subseteq p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \ldots \cup B_m) \). This completes the proof.

Combining Proposition 5.4, 5.5 and 5.6, we have the following theorem:

**Theorem 5.7.** Let \( A \) be a nonempty finite \( P_\gamma \)-open set and \( A_k \) a minimal \( P_\gamma \)-open set in \( A \) for each \( k \in \{ 1, 2, \ldots, m \} \). Then the following three conditions are equivalent:
Theorem 5.9. Then we have the result. □

Let \( P \) there exists finite

Proposition 4.1, there exists the class

Theorem 5.8. \( \{ \) of all minimal \( P \) -open sets in \( A \).

Then by Theorem 5.7, we have \( \gamma \) is pre regular.

Suppose that \( \phi \neq A \) is a finite \( P \gamma \)-open set and \( \{ A_1, A_2, \ldots, A_m \} \) is a class of all minimal \( P \gamma \)-open sets in \( A \) such that for each \( k \in \{ 1, 2, \ldots, m \} \), \( y_k \in A_k \). Then by Theorem 5.7, it is clear that \( \{ y_1, y_2, \ldots, y_m \} \) is a pre \( P \gamma \)-open set.

**Theorem 5.8.** Let \( A \) be a nonempty finite \( P \gamma \)-open set and \( \{ A_1, A_2, \ldots, A_m \} \) is a class of all minimal \( P \gamma \)-open sets in \( A \). Let \( B \) be any subset of \( A \setminus (A_1 \cup A_2 \cup \ldots \cup A_m) \) and \( B_k \) be any nonempty subset of \( A_k \) for each \( k \in \{ 1, 2, \ldots, m \} \). Then \( B \cup B_1 \cup B_2 \cup \ldots \cup B_m \) is a pre \( P \gamma \)-open set.

**Proof.** By Theorem 5.7, we have

\[
A \subseteq p_\gamma \text{Cl}(B_1 \cup B_2 \cup \ldots \cup B_m) \subseteq p_\gamma \text{Cl}(B \cup B_1 \cup B_2 \cup \ldots \cup B_m).
\]

Since \( A \) is a \( P \gamma \)-open set, then we have

\[
B \cup B_1 \cup B_2 \cup \ldots \cup B_m \subseteq A = p_\gamma \text{Int}(A) \subseteq p_\gamma \text{Int}(p_\gamma \text{Cl}(B \cup B_1 \cup B_2 \cup \ldots \cup B_m)).
\]

Then we have the result. □

**Theorem 5.9.** Let \( X \) be a \( P \gamma \)-locally finite space. If a minimal \( P \gamma \)-open set \( A \subseteq X \) has more than one element, then \( X \) is a pre \( P \gamma \)-Hausdorff space, where \( \gamma \) is pre regular.

**Proof.** Let \( x, y \in X \) such that \( x \neq y \). Since \( X \) is a \( P \gamma \)-locally finite space, there exists finite \( P \gamma \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \). By Proposition 4.1, there exists the class \( \{ U_1, U_2, \ldots, U_n \} \) of all minimal \( P \gamma \)-open sets in \( U \) and the class \( \{ V_1, V_2, \ldots, V_m \} \) of all minimal \( P \gamma \)-open sets in \( V \). We consider three possibilities:

1. If there exists \( i \) of \( \{ 1, 2, \ldots, n \} \) and \( j \) of \( \{ 1, 2, \ldots, m \} \) such that \( x \in U_i \) and \( y \in V_j \), then by Theorem 3.15, \( \{ x \} \) and \( \{ y \} \) are disjoint pre \( P \gamma \)-open sets which contains \( x \) and \( y \), respectively.
2. If there exists \( i \) of \( \{ 1, 2, \ldots, n \} \) such that \( x \in U_i \) and \( y \notin V_j \) for any \( j \) of \( \{ 1, 2, \ldots, m \} \), then we find an element \( y_j \) of \( V_j \) for each \( j \) such that \( \{ x \} \) and \( \{ y, y_1, y_2, \ldots, y_m \} \) are pre \( P \gamma \)-open sets and \( \{ x \} \cap \{ y, y_1, y_2, \ldots, y_m \} = \emptyset \) by Theorems 3.15, 5.8 and the assumption.
3. If \( x \notin U_i \) for any \( i \) of \( \{ 1, 2, \ldots, n \} \) and \( y \notin V_j \) for any \( j \) of \( \{ 1, 2, \ldots, m \} \), then we find elements \( x_i \) of \( U_i \) and \( y_j \) of \( V_j \) for each \( i, j \) such that \( \{ x, x_1, x_2, \ldots, x_n \} \) and \( \{ y, y_1, y_2, \ldots, y_m \} \) are pre \( P \gamma \)-open sets and \( \{ x, x_1, x_2, \ldots, x_n \} \cap \{ y, y_1, y_2, \ldots, y_m \} = \emptyset \) by Theorem 5.8 and the assumption. Hence \( X \) is a pre \( P \gamma \)-Hausdorff space.
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Department of Mathematics, College of Science, University of Duhok, Kurdistan-Region, Iraq

E-mail address: aliasbkhalaf@gmail.com

Department of Mathematics, Faculty of Science, University of Zakho, Kurdistan-Region, Iraq

E-mail address: hariwan_math@yahoo.com