FOUR-DIMENSIONAL MATRIX TRANSFORMATION AND A–STATISTICAL FUZZY KOROVKIN TYPE APPROXIMATION

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Abstract. In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using A-statistical convergence for four-dimensional summability matrices. Also we obtain rates of A-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

1. Introduction

Anastassiou [3] first introduced the fuzzy analogue of the classical Korovkin theory (see also [1], [2], [4], [10]). Recently, some statistical fuzzy approximation theorems have been obtained by using the concept of statistical convergence (see, [5], [8]). In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using A-statistical convergence for four-dimensional summability matrices. Then, we construct an example such that our new approximation result works but its classical case does not work. Also we obtain rates of A-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

We now recall some basic definitions and notations used in the paper.

A fuzzy number is a function \( \mu : \mathbb{R} \to [0,1] \), which is normal, convex, upper semi-continuous and the closure of the set \( \text{supp}(\mu) \) is compact, where \( \text{supp}(\mu) := \{ x \in \mathbb{R} : \mu(x) > 0 \} \). The set of all fuzzy numbers are denoted by \( \mathbb{R}_F \).

Let
\[
[\mu]^0 = \{ x \in \mathbb{R} : \mu(x) > 0 \} \quad \text{and} \quad [\mu]^r = \{ x \in \mathbb{R} : \mu(x) > r \}, \quad (0 < r < 1).
\]
Then, it is well-known [11] that, for each \( r \in [0,1] \), the set \( [\mu]^r \) is a closed and bounded interval of \( \mathbb{R} \). For any \( u, v \in \mathbb{R}_F \) and \( \lambda \in \mathbb{R} \), it is possible to define uniquely the sum \( u \oplus v \) and the product \( \lambda \odot u \) as follows:
\[
[u \oplus v]^r = [u]^r + [v]^r \quad \text{and} \quad [\lambda \odot u]^r = \lambda [u]^r, \quad (0 \leq r \leq 1).
\]

Now denote the interval \( [u]^r \) by \([u_-, u_+]^r\), where \( u_-(r) \leq u_+(r) \) and \( u_-(r), u_+(r) \in \mathbb{R} \) for \( r \in [0,1] \). Then, for \( u, v \in \mathbb{R}_F \), define
\[
u \leq v \iff u_-(r) \leq v_-(r) \leq u_+(r) \leq v_+(r) \text{ for all } 0 \leq r \leq 1.
\]
Define also the following metric \( D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \) by
\[
D(u, v) = \sup_{r \in [0,1]} \max \left\{ \left| u_-(r) - v_-(r) \right|, \left| u_+(r) - v_+(r) \right| \right\}.
\]

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Hence, \((\mathbb{R}^2, D)\) is a complete metric space\([18]\).

A double sequence \(x = \{x_{m,n}\}, \ m,n \in \mathbb{N}\), is convergent in Pringsheim’s sense if, for every \(\varepsilon > 0\), there exists \(N = N(\varepsilon) \in \mathbb{N}\) such that \(|x_{m,n} - L| < \varepsilon\) whenever \(m,n > N\). Then, \(L\) is called the Pringsheim limit of \(x\) and is denoted by \(P\lim_{m,n} x_{m,n} = L\) (see \([16]\)). In this case, we say that \(x = \{x_{m,n}\}\) is “\(P\)-convergent to \(L\)”. Also, if there exists a positive number \(M\) such that \(|x_{m,n}| \leq M\) for all \((m,n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}\), then \(x = \{x_{m,n}\}\) is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded. A double sequence \(x = \{x_{m,n}\}\) is said to be non-increasing in Pringsheim’s sense if, for all \((m,n)\) \(\in \mathbb{N}^2\), \(x_{m+1,n+1} \leq x_{m,n}\).

Now let \(A = [a_{j,k,m,n}]\), \(j,k,m,n \in \mathbb{N}\), be a four-dimensional summability matrix. For a given double sequence \(x = \{x_{m,n}\}\), the \(A\)-transform of \(x\), denoted by \(Ax : = \{(Ax)_{j,k}\}\), is given by

\[
(Ax)_{j,k} = \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j,k \in \mathbb{N},
\]

provided the double series converges in Pringsheim’s sense for every \((j,k) \in \mathbb{N}^2\). In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, \([13]\)). In 1926, Robison \([17]\) presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double \(P\)-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, \(RH\)-regularity (see, \([12]\), \([17]\)).

Recall that a four dimensional matrix \(A = [a_{j,k,m,n}]\) is said to be \(RH\)-regular if it maps every bounded \(P\)-convergent sequence into a \(P\)-convergent sequence with the same \(P\)-limit. The Robison-Hamilton conditions state that a four dimensional matrix \(A = [a_{j,k,m,n}]\) is \(RH\)-regular if and only if

\[
\begin{align*}
(i) \quad & P \lim_{j,k} a_{j,k,m,n} = 0 \text{ for each } (m,n) \in \mathbb{N}^2, \\
(ii) \quad & P \lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} = 1, \\
(iii) \quad & P \lim_{j,k} \sum_{m \in \mathbb{N}} |a_{j,k,m,n}| = 0 \text{ for each } n \in \mathbb{N}, \\
(iv) \quad & P \lim_{j,k} \sum_{n \in \mathbb{N}} |a_{j,k,m,n}| = 0 \text{ for each } m \in \mathbb{N}, \\
(v) \quad & \sum_{(m,n) \in \mathbb{N}^2} |a_{j,k,m,n}| \text{ is } P\text{-convergent } (j,k) \in \mathbb{N}^2, \\
(vi) \quad & \text{there exist finite positive integers } A \text{ and } B \text{ such that } \sum_{m,n>B} |a_{j,k,m,n}| < A \text{ holds for every } (j,k) \in \mathbb{N}^2.
\end{align*}
\]
Now let $A = [a_{j,k,m,n}]$ be a non-negative $RH$-regular summability matrix, and let $K \subseteq \mathbb{N}^2$. Then, a double sequence $\{x_{m,n}\}$ of fuzzy numbers is said to be $A$-statistically convergent to a fuzzy number $L \in \mathbb{R}_F$ if, for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m, n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\}.$$

In this case we write $st^2(A) - \lim_{m,n} x_{m,n} = L$.

We should note that if we take $A = C(1; 1) := [c_{j,k,m,n}]$, the double Cesáro matrix, defined by

$$c_{j,k,m,n} = \begin{cases} \frac{1}{j}, & \text{if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0, & \text{otherwise}, \end{cases}$$

then $C(1; 1)$–statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [14], [15]. Finally, if we replace the matrix $A$ by the identity matrix for four-dimensional matrices, then $A$–statistical convergence reduces to the Pringsheim convergence [16].

2. A-Statistical Fuzzy Korovkin Type Approximation

Let the real numbers $a; b; c; d$ so that $a < b, c < d$, and $U := [a; b] \times [c; d]$. Let $C(U)$ denote the space of all real valued continuous functions on $U$ endowed with the supremum norm

$$\|f\| = \sup_{(x,y) \in U} |f(x,y)|, \ (f \in C(U)).$$

Assume that $f : U \to \mathbb{R}_F$ be a fuzzy number valued function. Then $f$ is said to be fuzzy continuous at $x_0 := (x_0, y_0) \in U$ whenever $P - \lim_{m,n} x_{m,n} = x_0$, then $P - \lim D(f(x_{m,n}), f(x_0)) = 0$. If it is fuzzy continuous at every point $(x, y) \in U$, we say that $f$ is fuzzy continuous on $U$. The set of all fuzzy continuous functions on $U$ is denoted by $C_F(U)$. Note that $C_F(U)$ is a vector space. Now let $L : C_F(U) \to C_F(U)$ be an operator. Then $L$ is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$ having the same sign and for every $f_1, f_2 \in C_F(U)$, and $(x, y) \in U$,

$$L(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2 ; x, y) = \lambda_1 \odot L(f_1 ; x, y) \oplus \lambda_2 \odot L(f_2 ; x, y)$$

holds. Also $L$ is called fuzzy positive linear operator if it is fuzzy linear and, the condition $L(f ; x, y) \leq L(g ; x, y)$ is satisfied for any $f, g \in C_F(U)$ and all $(x, y) \in U$ with $f(x, y) \leq g(x, y)$. Also, if $f, g : U \to \mathbb{R}_F$ are fuzzy number valued functions, then the distance between $f$ and $g$ is given by

$$D^*(f, g) = \sup_{(x,y) \in U} \sup_{r \in [0,1]} \max \left\{ \left| f_+^{(r)} - g_+^{(r)} \right|, \left| f_-^{(r)} - g_-^{(r)} \right| \right\}$$

(see for details, [1], [2], [3], [4], [9], [10]). Throughout the paper we use the test functions given by

$$f_0(x, y) = 1, \ f_1(x, y) = x, \ f_2(x, y) = y, \ f_3(x, y) = x^2 + y^2.$$
Theorem 2.1. Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{L_{m,n}\}_{(m,n)\in \mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_F(U)$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_{m,n}\}_{(m,n)\in \mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property

$$(2.1) \quad \{L_{m,n}(f; x, y)\}_{i=\pm} = \tilde{L}_{m,n}(f_{i}; x, y)$$

for all $(x, y) \in U$, $r \in [0, 1]$, $(m, n) \in \mathbb{N}^2$ and $f \in C_F(U)$. Assume further that

$$(2.2) \quad s_{\ell^2(A)} - \lim_{m,n \to \infty} \|\tilde{L}_{m,n}(f_i) - f_i\| = 0 \quad \text{for each} \quad i = 0, 1, 2, 3.$$

Then, for all $f \in C_F(U)$, we have

$$s_{\ell^2(A)} - \lim_{m,n \to \infty} D^*(L_{m,n}(f), f) = 0.$$

Proof. Let $f \in C_F(U)$, $(x, y) \in U$ and $r \in [0, 1]$. By the hypothesis, since $f_{i}^{(r)} \in C(U)$, we can write, for every $\varepsilon > 0$, that there exists a number $\delta > 0$ such that $|f_{\pm}^{(r)}(u, v) - f_{\pm}^{(r)}(x, y)| < \varepsilon$ holds for every $(u, v) \in U$ satisfying $|u - x| < \delta$ and $v - y| < \delta$. Then we immediately get for all $(u, v) \in U$, that

$$|f_{\pm}^{(r)}(u, v) - f_{\pm}^{(r)}(x, y)\| \leq \varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2} \left\{ (u-x)^2 + (v-y)^2 \right\},$$

where $M_{\pm}^{(r)} := \|f_{\pm}^{(r)}\|$. Now, using the linearity and the positivity of the operators $\tilde{L}_{m,n}$, we have, for each $(m, n) \in \mathbb{N}^2$, that

$$\left| \tilde{L}_{m,n}(f_{\pm}^{(r)}; x, y) - f_{\pm}^{(r)}(x, y) \right|$$

$$\leq \tilde{L}_{m,n} \left| f_{\pm}^{(r)}(u, v) - f_{\pm}^{(r)}(x, y) \right| \pm M_{\pm}^{(r)} \left| \tilde{L}_{m,n}(f_0; x, y) - f_0(x, y) \right|$$

$$\leq \tilde{L}_{m,n} \left( \varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2} \left\{ (u-x)^2 + (v-y)^2 \right\} \right) \pm M_{\pm}^{(r)} \left| \tilde{L}_{m,n}(f_0; x, y) - f_0(x, y) \right|$$

$$\leq \varepsilon + \left( \varepsilon + M_{\pm}^{(r)} \right) \left| \tilde{L}_{m,n}(f_0; x, y) - f_0(x, y) \right| + \frac{2M_{\pm}^{(r)}}{\delta^2} \left| \tilde{L}_{m,n}(f_0; x, y) - f_0(x, y) \right|$$
\[
\begin{align*}
\leq & \varepsilon + \left( \varepsilon + M_{+}^{(r)} \right) \left| \tilde{L}_{m,n} (f_0; x, y) - f_0 (x, y) \right| + \frac{2M_{+}^{(r)}}{\delta^2} \left\{ \left| \tilde{L}_{m,n} (f_3; x, y) - f_3 (x, y) \right| \\
& + 2x^2 \left| \tilde{L}_{m,n} (f_1; x, y) - f_1 (x, y) \right| + 2y^2 \left| \tilde{L}_{m,n} (f_2; x, y) - f_2 (x, y) \right| + \\
& + (x^2 + y^2) \left| \tilde{L}_{m,n} (f_0; x, y) - f_0 (x, y) \right| \right\} \\
& \leq \varepsilon + \left( \varepsilon + M_{+}^{(r)} + \frac{2M_{+}^{(r)}}{\delta^2} (x^2 + y^2) \right) \left| \tilde{L}_{m,n} (f_0; x, y) - f_0 (x, y) \right| \\
& + \frac{4M_{+}^{(r)}}{\delta^2} x^2 \left| \tilde{L}_{m,n} (f_1; x, y) - f_1 (x, y) \right| + \frac{4M_{+}^{(r)}}{\delta^2} y^2 \left| \tilde{L}_{m,n} (f_2; x, y) - f_2 (x, y) \right| \\
& + \frac{2M_{+}^{(r)}}{\delta^2} \left| \tilde{L}_{m,n} (f_3; x, y) - f_3 (x, y) \right| \\
& \leq \varepsilon + K_{+}^{(r)} (\varepsilon) \left\{ \left| \tilde{L}_{m,n} (f_0; x, y) - f_0 (x, y) \right| + \left| \tilde{L}_{m,n} (f_1; x, y) - f_1 (x, y) \right| \\
& + \left| \tilde{L}_{m,n} (f_2; x, y) - f_2 (x, y) \right| + \left| \tilde{L}_{m,n} (f_3; x, y) - f_3 (x, y) \right| \right\} \\
\end{align*}
\]

where \( K_{+}^{(r)} (\varepsilon) := \max \left\{ \varepsilon + M_{+}^{(r)} + \frac{2M_{+}^{(r)}}{\delta^2} (A^2 + B^2), \frac{4M_{+}^{(r)}}{\delta^2} A, \frac{4M_{+}^{(r)}}{\delta^2} B, \frac{2M_{+}^{(r)}}{\delta^2} \right\} \), \( A := \max \{|a|, |b|\}, B := \max \{|c|, |d|\} \). Also taking supremum over \((x, y) \in U\), the above inequality implies that

\[(2.3) \quad \left| \tilde{L}_{m,n} (f_+^{(r)}) - f_+^{(r)} \right| \leq \varepsilon + K_{+}^{(r)} (\varepsilon) \left\{ \left| \tilde{L}_{m,n} (f_0) - f_0 \right| + \left| \tilde{L}_{m,n} (f_1) - f_1 \right| \\
+ \left| \tilde{L}_{m,n} (f_2) - f_2 \right| + \left| \tilde{L}_{m,n} (f_3) - f_3 \right| \right\}. \]

Now, it follows from (2.1) that

\[
D^a (L_{m,n} (f), f) = \sup_{(x, y) \in U} D (L_{m,n} (f; x, y), f (x, y)) \\
= \sup_{(x, y) \in U} \sup_{r \in [0,1]} \max \left\{ \left| \tilde{L}_{m,n} (f^{(r)}; x, y) - f^{(r)} (x, y) \right|, \left| \tilde{L}_{m,n} (f_+^{(r)}; x, y) - f_+^{(r)} (x, y) \right| \right\} \\
= \sup_{r \in [0,1]} \max \left\{ \left| \tilde{L}_{m,n} (f^{(r)}) - f^{(r)} \right|, \left| \tilde{L}_{m,n} (f_+^{(r)}) - f_+^{(r)} \right| \right\}.
\]
Combining the above equality with (2.3), we have

\[(2.4) \quad D^* \left( L_{m,n} (f), f \right) \leq \varepsilon + K(\varepsilon) \left\{ \left\| \tilde{L}_{m,n} (f_0) - f_0 \right\| + \left\| \tilde{L}_{m,n} (f_1) - f_1 \right\| + \left\| \tilde{L}_{m,n} (f_2) - f_2 \right\| + \left\| \tilde{L}_{m,n} (f_3) - f_3 \right\| \right\} \]

where \( K(\varepsilon) := \sup_{r \in [0,1]} \max \left\{ K^r_-(\varepsilon), K^r_+(\varepsilon) \right\} \).

Now, for a given \( r > 0 \), choose \( \varepsilon > 0 \) such that \( 0 < \varepsilon < r \), and also define the following sets:

\[ \begin{align*}
G & : = \left\{ (m,n) \in \mathbb{N}^2 : D^* \left( L_{m,n} (f), f \right) \geq r \right\}, \\
G_0 & : = \left\{ (m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n} (f_0) - f_0 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)} \right\}, \\
G_1 & : = \left\{ (m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n} (f_1) - f_1 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)} \right\}, \\
G_2 & : = \left\{ (m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n} (f_2) - f_2 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)} \right\}, \\
G_3 & : = \left\{ (m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n} (f_3) - f_3 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)} \right\}.
\end{align*} \]

Then inequality (2.4) gives

\[
G \subseteq G_0 \cup G_1 \cup G_2 \cup G_3
\]

which guarantees that, for each \((j, k) \in \mathbb{N}^2\)

\[
(2.5) \quad \sum_{(m,n) \in G} a_{j,k,m,n} \leq \sum_{(m,n) \in G_0} a_{j,k,m,n} + \sum_{(m,n) \in G_1} a_{j,k,m,n} + \sum_{(m,n) \in G_2} a_{j,k,m,n} + \sum_{(m,n) \in G_3} a_{j,k,m,n}.
\]

If we take as \( j, k \to \infty \) on the both sides of inequality (2.5) and use the hypothesis (2.2), we immediately see that

\[
\lim_{j,k} \sum_{(m,n) \in G} a_{j,k,m,n} = 0
\]

whence the result.

If \( A = I \), the identity matrix, then we obtain the following new fuzzy Korovkin theorem in Pringsheim’s sense.

**Theorem 2.2.** Let \( \{ L_{m,n} \}_{(m,n) \in \mathbb{N}^2} \) be a double sequence of fuzzy positive linear operators from \( C_F (U) \) into itself. Assume that there exists a corresponding sequence \( \{ \tilde{L}_{m,n} \}_{(m,n) \in \mathbb{N}^2} \) of positive linear operators from \( C(U) \) into itself with the property (2.1). Assume further that

\[
P - \lim_{m,n \to \infty} \left\| \tilde{L}_{m,n} (f_i) - f_i \right\| = 0 \quad \text{for each} \quad i = 0, 1, 2, 3.
\]

Then, for all \( f \in C_F (U) \), we have

\[
P - \lim_{m,n \to \infty} D^* \left( L_{m,n} (f), f \right) = 0.
\]
We now show that our result Theorem 2.1 stronger than its classical (Theorem 2.2) version.

**Example 2.1.** Take \( A = C(1, 1) := [c_{j, k, m, n}] \), the double Cesáro matrix, and define the double sequence \( \{u_{m, n}\} \) by

\[
u_{m, n} = \begin{cases} \sqrt{mn}, & \text{if } m \text{ and } n \text{ are square,} \\ 0, & \text{otherwise.} \end{cases}
\]

We observe that, \( \text{st}_{C(1,1)}^{(2)} \lim_{m, n \to \infty} u_{m, n} = 0 \). But \( \{u_{m, n}\} \) is neither \( P \)-convergent nor bounded. Then consider the Fuzzy Bernstein-type polynomials as follows:

\[
B_{m,n}^{(2)} (f; x, y) = (1 + u_{m, n}) \sum_{s=0}^{m} \sum_{t=0}^{n} \left( \frac{m}{s} \right) \left( \frac{n}{t} \right) x^s y^t (1 - x)^{m-s} (1 - y)^{n-t} f \left( \frac{s}{m}, \frac{t}{n} \right),
\]

where \( f \in C_x(U) \), \((x, y) \in U \), \((m, n) \in \mathbb{N}^2 \). In this case, we write

\[
\left\{ B_{m,n}^{(r)} (f; x, y) \right\}_{r=1}^{(r)} = \tilde{B}_{m,n} \left( f_{r}^{(r)} ; x \right)
\]

\[\quad = (1 + u_{m, n}) \sum_{s=0}^{m} \sum_{t=0}^{n} \left( \frac{m}{s} \right) \left( \frac{n}{t} \right) x^s y^t (1 - x)^{m-s} (1 - y)^{n-t} f_{r}^{(r)} \left( \frac{s}{m}, \frac{t}{n} \right),\]

where \( f_{r}^{(r)} \in C(U) \). Then, we get

\[
\begin{align*}
\tilde{B}_{m,n} (f_0; x) &= (1 + u_{m, n}) f_0 (x, y), \\
\tilde{B}_{m,n} (f_1; x) &= (1 + u_{m, n}) f_1 (x, y), \\
\tilde{B}_{m,n} (f_2; x) &= (1 + u_{m, n}) f_2 (x, y), \\
\tilde{B}_{m,n} (f_3; x) &= (1 + u_{m, n}) \left( f_3 (x, y) + \frac{x^2}{m} + \frac{y^2}{n} \right).
\end{align*}
\]

So we conclude that

\[
\text{st}_{C(1,1)}^{(2)} \lim_{m, n \to \infty} \left\| \tilde{B}_{m,n} (f_i) - f_i \right\| = 0 \quad \text{for each } i = 0, 1, 2, 3.
\]

By Theorem 2.1, we obtain for all \( f \in C_x(U) \), that

\[
\text{st}_{C(1,1)}^{(2)} \lim_{m, n \to \infty} D^* \left( B_{m,n}^{(r)} (f) , f \right) = 0.
\]

However, since the sequence \( \{u_{m, n}\} \) is not convergent (in the Pringsheim’s sense), we conclude that Theorem 2.2 do not work for the operators \( \left\{ B_{m,n}^{(r)} (f; x, y) \right\} \) in (2.6) while our Theorem 2.1 still works.

### 3. A-Statistical Fuzzy Rates

Various ways of defining rates of convergence in the \( A \)-statistical sense for two-dimensional summability matrices were introduced in [7]. In a similar way, we obtain fuzzy approximation theorems based on \( A \)-statistical rates for four-dimensional summability matrices.
Definition 3.1. Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ be a positive non-increasing double sequence. A double sequence $x = \{x_{m,n}\}$ is $A$-statistically convergent to a fuzzy number $L$ with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k \to \infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\}.$$

In this case, we write $D(x_{m,n}, L) = st^2_{(A)} - o(\alpha_{m,n})$ as $m, n \to \infty$.

Definition 3.2. Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 3.1. Then, a double sequence $x = \{x_{m,n}\}$ is $A$-statistically convergent to a fuzzy number $L$ with the rate of $o_m(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k \to \infty} \sum_{(m,n) \in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon \alpha_{m,n}\}.$$

In this case, we write $D(x_{m,n}, L) = st^2_{(A)} - o_m(\alpha_{m,n})$ as $m, n \to \infty$.

Note that the rate of convergence given by Definition 3.1 is more controlled by the entries of the summability matrices rather than the terms of the sequence $x = \{x_{m,n}\}$. However, according to the statistical rate given by Definition 3.2, the rate is mainly controlled by the terms of the fuzzy sequence $x = \{x_{m,n}\}$.

Also, we can give the corresponding $A$-statistical rates of real sequence $\{x_{m,n}\}$.

Definition 3.3. [6] Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ be a positive non-increasing double sequence. A double sequence $x = \{x_{m,n}\}$ is $A$-statistically convergent to a number $L$ with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k \to \infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}.$$

In this case, we write $x_{m,n} - L = st^2_{(A)} - o(\alpha_{m,n})$ as $m, n \to \infty$.

Definition 3.4. [6] Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 3.3. Then, a double sequence $x = \{x_{m,n}\}$ is $A$-statistically convergent to a number $L$ with the rate of $o_m(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k \to \infty} \sum_{(m,n) \in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon \alpha_{m,n}\}.$$
In this case, we write
\[ x_{m,n} - L = st^2_{(A)} - o_{m,n}(a_{m,n}) \text{ as } m,n \to \infty. \]

Then we have the following.

**Theorem 3.1.** Let \( A = [a_{j,k,m,n}] \) be a non-negative RH-regular summability matrix and let \( \{L_{m,n}\}_{(m,n) \in \mathbb{N}^2} \) be a double sequence of fuzzy positive linear operators from \( C_{[0,1]}(U) \) into itself. Assume that there exists a corresponding sequence \( \{\tilde{L}_{m,n}\}_{(m,n) \in \mathbb{N}^2} \) of positive linear operators from \( C(U) \) into itself with the property (2.1). Assume further that \( \{\alpha_{i,m,n}\}_{(m,n) \in \mathbb{N}^2}, i = 0,1,2,3 \) are non-increasing sequences of positive real numbers. If, for each \( i = 0,1,2,3 \)

\[ \tag{3.1} \left\| \tilde{L}_{m,n}(f_i) - f_i \right\| = st^2_{(A)} - o(\alpha_{i,m,n}) \text{ as } m,n \to \infty \]

then, for all \( f \in C_{[0,1]}(U) \), we have

\[ \tag{3.2} D^*(L_{m,n}(f), f) = st^2_{(A)} - o(\gamma_{m,n}) \text{ as } m,n \to \infty \]

where \( \gamma_{m,n} := \max_{0 \leq i \leq 3} \{\alpha_{i,m,n}\} \) for every \((m,n) \in \mathbb{N}^2\).

**Proof.** Let \( f \in C_{[0,1]}(U) \), \((x,y) \in U\) and \( r \in [0,1] \). Then, we immediately see from Theorem 2.1's proof that, for every \( \varepsilon > 0 \),

\[ \tag{3.3} D^*(L_{m,n}(f), f) \leq \varepsilon + K(\varepsilon) \left\{ \left\| \tilde{L}_{m,n}(f_0) - f_0 \right\| + \left\| \tilde{L}_{m,n}(f_1) - f_1 \right\| + \left\| \tilde{L}_{m,n}(f_2) - f_2 \right\| + \left\| \tilde{L}_{m,n}(f_3) - f_3 \right\| \right\} \]

where \( K(\varepsilon) := \sup_{r \in [0,1]} \max_{0 \leq i \leq 3} \{K^{(r)}_-(\varepsilon), K^{(r)}_+(\varepsilon)\} \).

Now, for a given \( r > 0 \), choose \( \varepsilon > 0 \) such that \( 0 < \varepsilon < r \), and also define the following sets:

\[ G := \{(m,n) \in \mathbb{N}^2 : D^*(L_{m,n}(f), f) \geq r\}, \]

\[ G_0 := \{(m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n}(f_0) - f_0 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}, \]

\[ G_1 := \{(m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n}(f_1) - f_1 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}, \]

\[ G_2 := \{(m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n}(f_2) - f_2 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}, \]

\[ G_3 := \{(m,n) \in \mathbb{N}^2 : \left\| \tilde{L}_{m,n}(f_3) - f_3 \right\| \geq \frac{r - \varepsilon}{4K(\varepsilon)}\}. \]

Then inequality (3.3) gives

\[ G \subset G_0 \cup G_1 \cup G_2 \cup G_3 \]

which guarantees that, for each \((j,k) \in \mathbb{N}^2\)

\[ \sum_{(m,n) \in G} a_{j,k,m,n} \leq \sum_{i=0}^{3} \left( \sum_{(m,n) \in G_i} a_{j,k,m,n} \right). \]
Also, by the definition of \(\gamma_{m,n}(m,n)\in\mathbb{N}^2\), we have

\[
\frac{1}{\gamma_{j,k}(m,n)\in G} \sum_{a_{j,k,m,n}} \leq \sum_{i=0}^{3} \left( \frac{1}{\alpha_{i,j,k}(m,n)\in G} \sum_{a_{j,k,m,n}} \right).
\]

If we take as \(j,k\to\infty\) on both sides of inequality (3.4) and use the hypothesis (3.1), we immediately see that

\[
P - \lim_{j,k\to\infty} \frac{1}{\gamma_{j,k}(m,n)\in G} \sum_{a_{j,k,m,n}}
\]

which gives (3.2). So, the proof is completed.

We also give the next result.

**Theorem 3.2.** Let \(A = [a_{j,k,m,n}], \{\alpha_{i,m,n}\}_{(m,n)\in\mathbb{N}^2} (i = 0, 1, 2, 3), \{\gamma_{m,n}\}_{(m,n)\in\mathbb{N}^2}, \{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}\) and \(\{\tilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}\) be the same as in Theorem 3.1 with the property (2.1). If, for each \(i = 0, 1, 2, 3\)

\[
\|\tilde{L}_{m,n}(f_i) - f_i\| = s\|L_{m,n}(A)\| - o_{m,n}(\alpha_{i,m,n}) \quad \text{as} \ m,n\to\infty
\]

then, for all \(f \in C(x(U))\), we have

\[
D^* (L_{m,n}(f), f) = s\|L_{m,n}(A)\| - o_{m,n}(\gamma_{m,n}) \quad \text{as} \ m,n\to\infty.
\]

**Proof.** By (3.3), it is clear that, for any \(\varepsilon > 0\),

\[
D^* (L_{m,n}(f), f) \\
\leq \varepsilon \gamma_{m,n} + B(\varepsilon) \left\{ \|\tilde{L}_{m,n}(f_0) - f_0\| + \|\tilde{L}_{m,n}(f_1) - f_1\| \\
+ \|\tilde{L}_{m,n}(f_2) - f_2\| + \|\tilde{L}_{m,n}(f_3) - f_3\| \right\}
\]

holds for some \(B(\varepsilon) > 0\). Now, as in the proof of Theorem 3.1 for a given \(\varepsilon' > 0\), choosing \(\varepsilon > 0\) such that \(\varepsilon < \varepsilon'\). Now we define the following sets:

\[
E : = \{(m,n)\in\mathbb{N}^2 : D^* (L_{m,n}(f), f) \geq \varepsilon' \gamma_{m,n}\} ,
\]

\[
E_0 : = \{(m,n)\in\mathbb{N}^2 : \|\tilde{L}_{m,n}(f_0) - f_0\| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{0,m,n}\} ,
\]

\[
E_1 : = \{(m,n)\in\mathbb{N}^2 : \|\tilde{L}_{m,n}(f_1) - f_1\| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{1,m,n}\} ,
\]

\[
E_2 : = \{(m,n)\in\mathbb{N}^2 : \|\tilde{L}_{m,n}(f_2) - f_2\| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{2,m,n}\} ,
\]

\[
E_3 : = \{(m,n)\in\mathbb{N}^2 : \|\tilde{L}_{m,n}(f_3) - f_3\| \geq \frac{\varepsilon' - \varepsilon}{4B(\varepsilon)} \alpha_{3,m,n}\}.
\]

In this case, we claim that

\[
E \subset E_0 \cup E_1 \cup E_2 \cup E_3.
\]
Indeed, otherwise, there would be an element \((m, n) \in E\) but \((m, n) \notin E_0 \cup E_1 \cup E_2 \cup E_3\). So, we get
\[
(m, n) \notin E_0 \Rightarrow \|L_m,n (f_0) - f_0\| < \frac{(\varepsilon' - \varepsilon)}{4B(\varepsilon)} \alpha_{0,m,n},
\]
\[
(m, n) \notin E_1 \Rightarrow \|L_m,n (f_1) - f_1\| < \frac{(\varepsilon' - \varepsilon)}{4B(\varepsilon)} \alpha_{1,m,n},
\]
\[
(m, n) \notin E_2 \Rightarrow \|L_m,n (f_2) - f_2\| < \frac{(\varepsilon' - \varepsilon)}{4B(\varepsilon)} \alpha_{2,m,n},
\]
\[
(m, n) \notin E_3 \Rightarrow \|L_m,n (f_3) - f_3\| < \frac{(\varepsilon' - \varepsilon)}{4B(\varepsilon)} \alpha_{3,m,n}.
\]

By the definition of \(\{\gamma_{m,n}\}_{(m,n) \in \mathbb{N}^2}\), we immediately see that
\[
B(\varepsilon) \sum_{k=0}^{3} \|L_m,n (f_k) - f_k\| < (\varepsilon' - \varepsilon) \gamma_{m,n}.
\]

Since \((m, n) \in E\), we have \(D^* (L_m,n (f), f) \geq \varepsilon \gamma_{m,n}\), and hence, by (3.7),
\[
B(\varepsilon) \sum_{k=0}^{3} \|L_m,n (f_k) - f_k\| \geq (\varepsilon' - \varepsilon) \gamma_{m,n},
\]
which contradicts with (3.9). So, our claim (3.8) holds true. Now, it follows from (3.8) that
\[
\sum_{(m,n) \in E} a_{j,k,m,n} \leq \sum_{i=0}^{3} \left( \sum_{(m,n) \in E_i} a_{j,k,m,n} \right).
\]

Letting \(j, k \to \infty\) in (3.10) and using (3.5), we observe that
\[
P - \lim_{j,k \to \infty} \sum_{(m,n) \in E} a_{j,k,m,n}
\]
which means (3.6). The proof is completed. \qed

**Remark 3.1.** If \(\alpha_{i,m,n} \equiv 1\) for each \(i = 0, 1, 2, 3\), then Theorem 3.2 reduced to Theorem 2.1. Also, if \(A = I\), the identity matrix, \(\alpha_{i,m,n} \equiv 1\) for each \(i = 0, 1, 2, 3\), then Theorem 3.2 reduced to Theorem 2.2.

**References**


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