DEMONSTRATIO MATHEMATICA
Vol. XLVI No 4 2013

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HERMITE–HADAMARD INEQUALITIES FOR CONVEX SET-VALUED FUNCTIONS

Abstract. The following version of the weighted Hermite–Hadamard inequalities for set-valued functions is presented: Let Y be a Banach space and $F : [a,b] \to cl(Y)$ be a continuous set-valued function. If F is convex, then

$$F(x_\mu) \supset \frac{1}{\mu([a,b])} \int_a^b F(x)\, d\mu(x) \supset \frac{b-x_\mu}{b-a} F(a) + \frac{x_\mu - a}{b-a} F(b),$$

where μ is a Borel measure on $[a,b]$ and x_μ is the barycenter of μ on $[a,b]$. The converse result is also given.

1. Introduction

It is well known that if a function $f : I \to \mathbb{R}$ is convex, that is

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \quad x, y \in I,\ t \in [0,1],$$

then it satisfies the following Hermite–Hadamard double inequality

$$(1) \qquad f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x} \int_x^y f(t)\, dt \le \frac{f(x)+f(y)}{2}, \quad x, y \in I,\ x < y.$$

Moreover, for continuous functions f, the validity of the left or the right-hand side inequality in (1) is equivalent to the convexity of f (cf. e.g. [1], [2], [5], [6], [8]).

The purpose of this note is to prove a set-valued counterpart of the weighted version of the above Hermite–Hadamard inequality. Our main theorem generalizes some earlier results of this type obtained by E. Sadowska [11] and B. Piątek [9]. As a consequence of the theorem, we obtain a set-valued counterpart of the classical Fejér inequality. We present also a converse of the Hermite–Hadamard theorem for set-valued functions.

2010 *Mathematics Subject Classification*: Primary 26A51; Secondary 54C60, 28B20, 39B62.

Key words and phrases: convex set-valued function, Hermite–Hadamard inequality.

Let Y be a Banach space and $I \subset \mathbb{R}$ be an interval. We denote by $n(Y)$ and $cl(Y)$ the families of all non-empty and non-empty closed subsets of Y, respectively. A set-valued function $F : I \to n(Y)$ is called *convex* if

$$F(tx_1 + (1-t)x_2) \supset tF(x_1) + (1-t)F(x_2), \tag{2}$$

for all $x_1, x_2 \in I$ and $t \in [0,1]$. $F : I \to n(Y)$ is said to be *continuous* at a point x_0 if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in $\mathbb{R}$ such that

$$F(x) \subset F(x_0) + V \quad \text{and} \quad F(x_0) \subset F(x) + V,$$

for all $x \in (x_0 + U) \cap I$. A function $f : I \to Y$ is called *selection* of F if $f(x) \in F(x)$, for every $x \in I$. Given a Borel measure μ on I and $[a,b] \subset I$, we denote by $\int_a^b F(x)\,d\mu(x)$ the Aumann integral of F, i.e. the set of the integrals of all μ-integrable selections of F.

2. Main results

The weighted Hermite–Hadamard theorem for set-valued functions reads as follows.

Theorem 1. *Let $F : [a,b] \to cl(Y)$ be a continuous convex set-valued function and μ be a Borel measure on $[a,b]$ with $\mu([a,b]) > 0$. Then*

$$F(x_\mu) \supset \frac{1}{\mu([a,b])} \int_a^b F(x)\,d\mu(x) \supset \frac{b - x_\mu}{b-a}\,F(a) + \frac{x_\mu - a}{b-a}\,F(b), \tag{3}$$

where $x_\mu = \frac{1}{\mu([a,b])} \int_a^b x\,d\mu(x)$ is the barycenter of μ on $[a,b]$.

In the proof of this theorem we will use the following lemma (cf. [4]).

Lemma 2. *If a set-valued function $F : [a,b] \to cl(\mathbb{R})$ is continuous and convex, then it has one of the following forms:*

a) $F(x) = [f_1(x), f_2(x)], \quad x \in [a,b]$;
b) $F(x) = [f_1(x), +\infty), \quad x \in [a,b]$;
c) $F(x) = (-\infty, f_2(x)], \quad x \in [a,b]$;
d) $F(x) = (-\infty, +\infty), \quad x \in [a,b]$,

where $f_1 : [a,b] \to \mathbb{R}$ is a convex continuous function and $f_2 : [a,b] \to \mathbb{R}$ is a concave continuous function.

Proof. By the convexity of F it follows that if $F(x_0)$ is bounded from below (above) for some $x_0 \in [a,b]$, then $F(x)$ is bounded from below (above) for every $x \in [a,b]$ (in the case $x_0 \in \{a,b\}$ we use additionally the continuity of F at x_0). Now, it is enough to define

$$f_1(x) = \begin{cases} \inf F(x), & \text{if } F(x) \text{ is bounded from below} \\ -\infty, & \text{otherwise} \end{cases}$$

and

$$f_2(x) = \begin{cases} \sup F(x), & \text{if } F(x) \text{ is bounded from above} \\ +\infty, & \text{otherwise} \end{cases}$$

and use the fact that the values of F are closed and convex. ∎

Proof of Theorem 1. The proof of the left-hand side inclusion is divided into two steps. First, we assume that $Y = \mathbb{R}$. Then, by Lemma 1, F has one of the forms a), b), c), d). Assume that $F(x) = [f_1(x), f_2(x)]$, $x \in [a, b]$ (the proof in the remaining cases is similar). Let h be a μ-integrable selection of F. Then, by the weighted Hermite–Hadamard inequality (see [2], [6]), we have

$$f_1(x_\mu) \le \frac{1}{\mu([a,b])} \int_a^b f_1(x)\, d\mu(x) \le \frac{1}{\mu([a,b])} \int_a^b h(x)\, d\mu(x)$$

and

$$f_2(x_\mu) \ge \frac{1}{\mu([a,b])} \int_a^b f_2(x)\, d\mu(x) \ge \frac{1}{\mu([a,b])} \int_a^b h(x)\, d\mu(x).$$

Hence

$$F(x_\mu) \ni \frac{1}{\mu([a,b])} \int_a^b h(x)\, d\mu(x),$$

and, consequently,

$$F(x_\mu) \supset \frac{1}{\mu([a,b])} \int_a^b F(x)\, d\mu(x).$$

Now, assume that Y is an arbitrary Banach space. Take a continuous linear functional $y^* \in Y^*$ and consider the set-valued function $\overline{y^* \circ F}$ defined by $\overline{y^* \circ F}(x) = \overline{y^*(F(x))}$, $x \in [a, b]$. This set-valued function is convex and continuous and its values are closed subsets of $\mathbb{R}$. Therefore, by the previous step,

$$\overline{y^*(F(x_\mu))} \supset \frac{1}{\mu([a,b])} \int_a^b \overline{y^*(F(x))}\, d\mu(x). \tag{4}$$

Let h be an arbitrary μ-integrable selection of F. Then $y^* \circ h$ is a μ-integrable selection of $y^* \circ F$. Therefore, by (4) and the fact that $\int_a^b y^* \circ h(x)\, d\mu(x) = y^*(\int_a^b h(x)\, d\mu(x))$, we get

$$\overline{y^*(F(x_\mu))} \ni y^*\left(\frac{1}{\mu([a,b])} \int_a^b h(x)\, d\mu(x) \right).$$

Since this condition holds for every $y^* \in Y^*$, by the separation theorem (cf.

[10], Corollary 2.5.11), we obtain

$$F(x_\mu) \ni \frac{1}{\mu([a,b])} \int_a^b h(x)\, d\mu(x).$$

Thus

$$F(x_\mu) \supset \frac{1}{\mu([a,b])} \int_a^b F(x)\, d\mu(x),$$

which proves the left-hand side inclusion in (3).

In order to prove the right-hand side inclusion in (3) take arbitrary

$$z = \frac{b - x_\mu}{b-a}\, u + \frac{x_\mu - a}{b-a}\, v,$$

where $u \in F(a)$ and $v \in F(b)$. Define

$$f(x) = \frac{b-x}{b-a}\, u + \frac{x-a}{b-a}\, v, \quad x \in [a,b].$$

By the convexity of F, we have

$$f(x) \in \frac{b-x}{b-a}\, F(a) + \frac{x-a}{b-a}\, F(b) \subset F(x),$$

which means that f is a selection of F. Moreover, being continuous, f is μ-integrable. Since

$$\begin{aligned}\int_a^b f(x)\, d\mu(x) &= \int_a^b \left(\frac{b-x}{b-a}\, u + \frac{x-a}{b-a}\, v\right) d\mu(x)\\ &= \mu([a,b])\left(\frac{b-x_\mu}{b-a}\, u + \frac{x_\mu - a}{b-a}\, v\right) = \mu([a,b])\, z,\end{aligned}$$

we get

$$\frac{1}{\mu([a,b])} \int_a^b F(x)\, d\mu(x) \ni z.$$

This shows that

$$\frac{1}{\mu([a,b])} \int_a^b F(x)\, d\mu(x) \supset \frac{b-x_\mu}{b-a}\, F(a) + \frac{x_\mu - a}{b-a}\, F(b)$$

and finishes the proof. ∎

Remark 3. In the particular case $\mu = \lambda$ (the Lebesgue measure), (3) reduces to

$$F\left(\frac{a+b}{2}\right) \supset \frac{1}{b-a} \int_a^b F(x)\, dx \supset \frac{F(a) + F(b)}{2}.$$

This result was obtained by E. Sadowska [11]. The second inclusion has been also proved, for the Hukuhara integral, by B. Piątek [9].

Remark 4. In [4], J. Matkowski and K. Nikodem proved the following multivalued version of the Jensen integral inequality: Let X, Y be Banach spaces and $D \subset X$ be an open convex set. If a set-valued function $F : D \to cl(Y)$ is continuous and convex, then for each normalized measure space (Ω, Σ, μ) and for all μ-integrable functions $\varphi : \Omega \to D$ such that $cl\, conv\, \varphi(\Omega) \subset D$, we have

$$F\Big(\int_\Omega \varphi\, d\mu\Big) \supset \int_\Omega F \circ \varphi\, d\mu.$$

Using this result, we can get immediately the left-hand side inclusion in (3) under the assumption that F is defined on an open interval $I \supset [a,b]$ (it is enough to take $\varphi(x) = x$, $x \in [a,b]$, and the normalized measure $\mu/\mu([a,b])$ on $[a,b]$). Note also that if the measure μ is atomless then the barycenter x_μ of μ on $[a,b]$ belongs to (a,b). In this case, the above Jensen inclusion remains true for $F : [a,b] \to cl(Y)$ and we can also obtain the left-hand side inclusion in (3) directly from it.

Let $g : [a,b] \to [0,\infty)$ be a *symmetric density function on* $[a,b]$ (that is, $g(a+b-x) = g(x)$, for all $x \in [a,b]$, and $\int_a^b g(x)\, dx = 1$). Taking μ in Theorem 1 such that $d\mu(x) = g(x)dx$ (i.e. g is the Radon–Nikodym derivative of μ with respect to the Lebesgue measure), we obtain the following set-valued counterpart of the classical Fejér inequality.

Corollary 5. *Let $F : [a,b] \to cl(Y)$ be a continuous convex set-valued function and $g : [a,b] \to [0,\infty)$ be a symmetric density function on $[a,b]$. Then*

$$F\Big(\frac{a+b}{2}\Big) \supset \int_a^b g(x)F(x)\, dx \supset \frac{F(a)+F(b)}{2}.$$

As a consequence of Theorem 1, we obtain also the following result.

Corollary 6. *Let $F : [a,b] \to cl(Y)$ be a continuous convex set-valued function. Then*

$$\frac{1}{b-a}\int_a^b F(x)\, dx \supset \frac{1}{2}\Big[\frac{F(a)+F(b)}{2} + F\Big(\frac{a+b}{2}\Big)\Big].$$

Proof. Applying the right-hand side inclusion in (3) on the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ for $\mu = \lambda$, we get

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} F(x)\, dx \supset \frac{1}{2}\Big[F(a) + F\Big(\frac{a+b}{2}\Big)\Big]$$

and

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} F(x)\,dx \supset \frac{1}{2}\left[F\left(\frac{a+b}{2}\right) + F(b)\right].$$

Summing up and taking into account the additivity of the integral, we obtain the conclusion. ∎

Remark 7. Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function and $F(x) = [f(x), +\infty)$, $x \in [a,b]$. Then, by Corollary 6, we get the following known inequality concerning convex functions (cf. [6], [8]):

$$\frac{1}{b-a}\int_{a}^{b} f(t)\,dt \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right].$$

3. The converse of Hermite–Hadamard theorem

It is known that if a continuous function $f : I \to R$ satisfies the left or the right-hand side inequality in (1), then it is convex. In this section, we present a set-valued counterpart of that result. In what follows, we assume that Y is a separable Banach space and denote by $bccl(Y)$ the family of all bounded convex closed and non-empty subsets of Y.

Theorem 8. *Let $F : I \to bccl(Y)$ be a continuous set-valued function such that*

$$F\left(\frac{a+b}{2}\right) \supset \frac{1}{b-a}\int_{a}^{b} F(x)\,dx, \quad a,b \in I,\ a<b \tag{5}$$

or

$$\frac{1}{b-a}\int_{a}^{b} F(x)\,dx \supset \frac{F(a)+F(b)}{2}, \quad a,b \in I,\ a<b. \tag{6}$$

Then F is convex.

Proof. We proceed by reductio ad absurdum. Suppose that F is not convex, i.e. (2) does not hold. Then there exist $t_0 \in (0,1)$, $x_1, x_2 \in I$ and $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ such that

$$z = t_0 y_1 + (1-t_0) y_2 \notin F(t_0 x_1 + (1-t_0) x_2).$$

Since the set $F(t_0 x_1 + (1-t_0)x_2)$ is convex and closed, by the separation theorem, there exists a continuous linear functional $y^* \in Y^*$ such that

$$y^*(z) > \sup\{y^*(y) : y \in F(t_0 x_1 + (1-t_0)x_2)\}. \tag{7}$$

Now, if F satisfies (5) then also

$$y^*\left(F\left(\frac{a+b}{2}\right)\right) \supset \frac{1}{b-a} y^*\left(\int_{a}^{b} F(x)\,dx\right), \quad a,b \in I,\ a<b. \tag{8}$$

Consider the function $f : I \to \mathbb{R}$ defined by $f(x) = \sup y^*(F(x))$, $x \in I$. Clearly, f is continuous and, in view of (8) and the fact that

$$\sup y^*\Big(\int_a^b F(x)\,dx\Big) = \int_a^b \sup y^*(F(x))\,dx$$

(see [3], Prop. 5.2), it satisfies

$$f\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a}\int_a^b f(x)\,dx, \quad a, b \in I,\ a < b.$$

Therefore f is concave.

Similarly, if F satisfies (6) then also

$$\frac{1}{b-a} y^*\Big(\int_a^b F(x)\,dx\Big) \supset \frac{y^*(F(a)) + y^*(F(b))}{2}, \quad a, b \in I,\ a < b.$$

From here

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq \frac{f(a)+f(b)}{2}, \quad a, b \in I,\ a < b$$

and, consequently, f is also concave. In both cases, by the concavity of f, we have

$$f(t_0x_1 + (1-t_0)x_2) \geq t_0 f(x_1) + (1-t_0)f(x_2).$$

Hence

$$\begin{aligned}\sup y^*(F(t_0x_1 + (1-t_0)x_2)) &\geq t_0 \sup y^*(F(x_1)) + (1-t_0)\sup y^*(F(x_2)) \\ &\geq t_0 y^*(y_1) + (1-t_0)y^*(y_2) = y^*(z).\end{aligned}$$

This contradicts (7) and finishes the proof. ∎

Remark 9. The fact that if F is continuous and satisfies condition (6) for the Hukuhara integral then it is convex, was proved by B. Piątek [9]. Note also that conditions (5) and (6) satisfied together imply the Jensen convexity of F, i.e.

$$F\left(\frac{a+b}{2}\right) \supset \frac{F(a)+F(b)}{2}, \quad a, b \in I,\ a < b.$$

Hence, under weak regularity assumptions, such as lower semicontinuity at a point, measurability, boundedness on a set with non-empty interior etc., it follows that F is convex (cf. e.g. [7] and the references therein).

Acknowledgements. The authors began to write this paper during the visit of the first author at the University of Bielsko-Biała, Poland. The first author is grateful to the Department of Mathematics and Computer Science for the hospitality and for financial support.

References

[1] M. Bessenyei, Zs. Páles, *Characterization of convexity via Hadamard's inequality*, Math. Inequal. Appl. 9(1) (2006), 53–62.

[2] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2002. (online: http://rgmia.vu.edu.au/monographs/).

[3] S. Hu, N. S. Papageorgiou, *Handbook of Multivalued Analysis.* Vol.I: *Theory*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1997.

[4] J. Matkowski, K. Nikodem, *An integral Jensen inequality for convex multifunctions*, Results Math. 26 (1994), 348–353.

[5] D. S. Mitrinović, L. B. Lacković, *Hermite and convexity*, Aequationes Math. 28 (1985), 229–232.

[6] C. P. Niculescu, L. -E. Persson, *Convex Functions and their Applications. A Contemporary Approach*, CMS Books in Mathematics vol. 23, Springer-Verlag, New York, 2006.

[7] K. Nikodem, *K-convex and K-concave set–valued functions*, Zeszyty Nauk. Politech. Łódz. Mat. 559; Rozprawy Mat. 114., Łódź, 1989.

[8] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Acad. Press, Inc., Boston, 1992.

[9] B. Piątek, *On convex and *-concave multifunctions*, Ann. Polon. Math. 86 (2005), 165–170.

[10] S. Rolewicz, *Functional Analysis and Control Theory. Linear Systems*, PWN - Polish Scientific Publishers & D. Reidel Publishing Company, Dordrecht/Boston/Lancaster/Tokyo, 1987.

[11] E. Sadowska, *Hadamard inequality and a refinement of Jensen inequality for set-valued functions*, Results Math. 32 (1997), 332–337.

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Received April 27, 2012.