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WEAKLY COMPATIBLE MAPS OF TYPE (A)
IN G-METRIC SPACES

Abstract. In this paper, we introduce the concept of compatible maps and weakly compatible maps of type (A) in G-metric spaces.

1. Introduction

In 1922, Banach proved fixed-point theorem (“Let \((X, d)\) be a complete metric space. If \(T\) satisfies \(d(Tx, Ty) \leq kd(x, y)\) for each \(x, y \in X\) where \(0 < k < 1\), then \(T\) has a unique fixed point in \(X\).”), which ensures under appropriate conditions, the existence and uniqueness of a fixed-point. This theorem had many applications, but suffers from one drawback—the definition requires that \(T\) be continuous throughout \(X\). Then there followed a flood of papers involving contractive definition that do not require the continuity of \(T\). This result was further generalized and extended in various ways by many authors ([1], [4], [5]). This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering.

In 1963, Gahler [3] introduced the concept of 2-metric spaces and claimed that a 2-metric is a generalization of the usual notion of a metric, but some authors proved that there is no relation between these two functions. It is clear that in 2-metric \(d(x, y, z)\) is to be taken as the area of the triangle with vertices at \(x, y\) and \(z\) in \(R^2\). However, Hsiao [2] showed that, for every contractive definition, with \(x_n = T^n x_0\), every orbit is linearly dependent, thus rendering fixed point theorems in such spaces trivial.

In 1992, Dhage [1] introduced the concept of \(D\)-metric space. The situation for a \(D\)-metric space is quite different from 2-metric spaces. Geometrically, a \(D\)-metric \(D(x, y, z)\) represent the perimeter of the triangle with vertices \(x, y\) and \(z\) in \(R^2\). Recently, Mustafa and Sims [6] showed that most of

\[\text{2000 Mathematics Subject Classification: 47H10, 54H25.}\]
\[\text{Key words and phrases: G-metric space, compatible maps of type (A), weakly compatible maps of type (A).}\]
the results concerning Dhage’s $D$-metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure, which they called $G$-metric space. For more details on $G$-metric spaces, one can referred to the papers [7]–[13].

Now we give preliminaries and basic definitions which are used throughout the paper.

In 2006, Mustafa and Sims [7] introduced the concept of $G$-metric spaces as follows:

**Definition 1.1.** [7] Let $X$ be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x,y,z) = 0$ if $x = y = z$,
(G2) $0 < G(x,x,y)$, for all $x,y \in X$ with $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$ with $z \neq y$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, (rectangle inequality),

then the function $G$ is called a generalized metric, or, more specifically a $G$-metric on $X$ and the pair $(X,G)$ is called a $G$-metric space.

**Definition 1.2.** [7] Let $(X,G)$ be a $G$-metric space, and let $\{x_n\}$ be a sequence of points in $X$, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n \to \infty} G(x,x_n,x_m) = 0$ and one says that sequence $\{x_n\}$ is $G$-convergent to $x$. Thus, that if $x_n \to x$ or $\lim_{n \to \infty} x_n \to x$ as $n \to \infty$ in a $G$-metric space $(X,G)$ then for each $\epsilon > 0$, there exists a positive integer $N$ such that $G(x,x_n,x_m) < \epsilon$ for all $m,n \geq N$.

Now we state some results from the papers ([7]–[9]) which are helpful for proving our main results.

**Proposition 1.1.** [7] Let $(X,G)$ be a $G$-metric space. Then the following are equivalent:

1. $\{x_n\}$ is $G$-convergent to $x$,
2. $G(x_n,x_n,x) \to 0$ as $n \to \infty$,
3. $G(x_n,x,x) \to 0$ as $n \to \infty$,
4. $G(x_m,x_n,x) \to 0$ as $m,n \to \infty$.

**Definition 1.3.** [7] Let $(X,G)$ be a $G$-metric space. A sequence $\{x_n\}$ is called $G$-Cauchy if, for each $\epsilon > 0$ there exists a positive integer $N$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $n,m,l \geq N$; i.e. if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to N$. 

Proposition 1.2. [7] If \((X, G)\) is a \(G\)-metric space then the following are equivalent:

1. the sequence \(\{x_n\}\) is \(G\)-Cauchy,
2. for each \(\epsilon > 0\), there exists a positive integer \(N\) such that \(G(x_n, x_m, x_m) < \epsilon\) for all \(n, m \geq N\).

Proposition 1.3. [7] Let \((X, G)\) be a \(G\)-metric space. Then the function \(G(x, y, z)\) is jointly continuous in all three of its variables.

Definition 1.4. [7] A \(G\)-metric space \((X, G)\) is called a symmetric \(G\)-metric space if \(G(x, y, y) = G(y, x, x)\) for all \(x, y \in X\).

Proposition 1.4. [7] Every \(G\)-metric space \((X, G)\) will defines a metric space \((X, d_G)\) by

1. \(d_G(x, y) = G(x, y, y) + G(y, x, x)\) for all \(x, y \in X\).

If \((X, G)\) is a symmetric \(G\)-metric space, then

2. \(d_G(x, y) = 2G(x, y, y)\) for all \(x, y \in X\).

However, if \((X, G)\) is not symmetric, then it follows from the \(G\)-metric properties that

3. \(3/2G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)\) for all \(x, y \in X\).

Definition 1.5. [7] A \(G\)-metric space \((X, G)\) is said to be \(G\)-complete if every \(G\)-Cauchy sequence in \((X, G)\) is \(G\)-convergent in \(X\).

Proposition 1.5. [7] A \(G\)-metric space \((X, G)\) is \(G\)-complete if and only if \((X, d_G)\) is a complete metric space.

Proposition 1.6. [7] Let \((X, G)\) be a \(G\)-metric space. Then, for any \(x, y, z, a \in X\) it follows that:

1. if \(G(x, y, z) = 0\), then \(x = y = z\),
2. \(G(x, y, z) \leq G(x, x, y) + G(x, x, z)\),
3. \(G(x, y, y) \leq 2G(y, x, x)\),
4. \(G(x, y, z) \leq G(x, a, z) + G(a, y, z)\),
5. \(G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))\),
6. \(G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))\).

Definition 1.6. [7] Let \((X, G)\) and \((X^0, G^0)\) be \(G\)-metric spaces and \(f : (X, G) \rightarrow (X^0, G^0)\) be a function, then \(f\) is said to be \(G\)-continuous at a point \(a \in X\) if and only if, given \(\epsilon > 0\) there exists \(\delta > 0\) such that \(x, y \in X\) and \(G(a, x, y) < \delta\) implies \(G^0(f(a), f(x), f(y)) < \epsilon\). A function \(f\) is \(G\)-continuous at \(X\) if and only if it is \(G\)-continuous at all \(a \in X\).
2. Main results

In 1998, Jungck and Rhoades [5] introduced the concept of weakly compatibility as follows:

**Definition 2.1.** [5] Let \((X, G)\) be a \(G\)-metric space, \(f\) and \(g\) be self maps on \(X\). A point \(x \in X\) is called a coincidence point of \(f\) and \(g\) iff \(fx = gx\). In this case, \(w = fx = gx\) is called a point of coincidence of \(f\) and \(g\).

**Definition 2.2.** [5] Two self-mappings \(S\) and \(T\) are said to be weakly compatible if they commute at coincidence points.

We introduce following definitions:

**Definition 2.3.** Let \(S\) and \(T\) be maps from a \(G\)-metric space \((X, G)\) into itself. The maps \(S\) and \(T\) are said to be compatible map if

\[
\lim_{n \to \infty} G(STx_n, TSx_n, TSx_n) = 0 \quad \text{or} \quad \lim_{n \to \infty} G(TSx_n, STx_n, STx_n) = 0
\]

whenever \(\{x_n\}\) is sequence in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} T x_n = t\) for some \(t \in X\).

**Definition 2.4.** Let \(S\) and \(T\) be maps from a \(G\)-metric space \((X, G)\) into itself. The maps \(S\) and \(T\) are said to be compatible map of type \((A)\) if

\[
\lim_{n \to \infty} G(TSx_n, SSx_n, SSx_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} G(STx_n, TTx_n, TTx_n) = 0
\]

whenever \(\{x_n\}\) is sequence in \(X\) such that \(\lim_{n \to \infty} S\{x_n\} = \lim_{n \to \infty} T x_n = t\) for some \(t \in X\).

**Definition 2.5.** Let \(S\) and \(T\) be maps from a \(G\)-metric space \((X, G)\) into itself. The maps \(S\) and \(T\) are said to be weakly compatible of type \((A)\) if

\[
\lim_{n \to \infty} G(TSx_n, SSx_n, SSx_n) \leq \lim_{n \to \infty} G(STx_n, SSx_n, SSx_n)
\]

and

\[
\lim_{n \to \infty} G(STx_n, TTx_n, TTx_n) \leq \lim_{n \to \infty} G(TSx_n, TTx_n, TTx_n)
\]

whenever \(\{x_n\}\) is sequence in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} T x_n = t\) for some \(t \in X\).

The following propositions show that Definitions 2.3 and 2.4 are equivalent under some conditions:

**Proposition 2.1.** Let \((X, G)\) be a \(G\)-metric space and let \(S, T : X \to X\) be \(G\)-continuous mappings. If \(S\) and \(T\) are compatible then they are compatible of type \((A)\).
Proof. Suppose $S$ and $T$ are compatible. Let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$. By (G5),
\[
G(TSx_n, SSx_n, SSx_n) \leq G(TSx_n, STx_n, STx_n) + G(STx_n, SSx_n, SSx_n).
\]

Since $S$ and $T$ are compatible and $S$ is $G$-continuous, we have
\[
\lim_{n \to \infty} G(TSx_n, SSx_n, SSx_n) = 0.
\]
Similarly, if $T$ is $G$-continuous, we have
\[
\lim_{n \to \infty} G(STx_n, TTx_n, TTx_n) = 0.
\]
Therefore, $S$ and $T$ are compatible of type (A).

Proposition 2.2. Let $(X, G)$ be a $G$-metric space and let $S, T : X \to X$ be compatible mappings of type (A). If both $S$ and $T$ are $G$-continuous, then $S$ and $T$ are compatible.

Proof. To show that $S$ and $T$ are compatible, suppose that $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$ then, as $T$ is $G$-continuous,
\[
\lim_{n \to \infty} TSx_n = \lim_{n \to \infty} TTx_n = Tt.
\]

By (G5),
\[
G(STx_n, TSx_n, TSx_n) \leq G(STx_n, TTx_n, TTx_n) + G(TTx_n, TSx_n, TSx_n).
\]

Since $S$ and $T$ are compatible mappings of type (A) and $T$ is $G$-continuous, we have $\lim_{n \to \infty} G(STx_n, TSx_n, TSx_n) = 0$. Similarly, if $S$ is $G$-continuous, we have $\lim_{n \to \infty} G(TSx_n, STx_n, STx_n) = 0$. Therefore, $S$ and $T$ are compatible.

From Propositions 2.1 and 2.2, we have:

Proposition 2.3. Let $S$ and $T$ be $G$-continuous mappings from a $G$-metric space $(X, G)$ into itself. Then $S$ and $T$ are compatible iff they are compatible of type (A).

The following propositions show that Definitions 2.4 and 2.5 are equivalent under some conditions:

Proposition 2.4. Every pair of compatible mappings of type (A) is weakly compatible of type (A).

Proof. Suppose that pair $\{S, T\}$ of maps is compatible of type (A), then we have
\[
0 = \lim_{n \to \infty} G(TSx_n, SSx_n, SSx_n) \leq \lim_{n \to \infty} G(STx_n, SSx_n, SSx_n).
\]
and
\[ 0 = \lim_{n \to \infty} G(STx_n, TTx_n, TTx_n) \leq \lim_{n \to \infty} G(TSx_n, TTx_n, TTx_n), \]
whenever \( \{x_n\} \) is sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \), which is always true. Therefore, \( S \) and \( T \) are weakly compatible of type \((A)\). ■

**Proposition 2.5.** Let \( S \) and \( T \) be \( G \)-continuous mappings of a \( G \)-metric space \((X,G)\) into itself. If \( S \) and \( T \) are weakly compatible of type \((A)\), then they are compatible of type \((A)\).

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). As \( S \) and \( T \) are \( G \)-continuous maps, therefore,

\[ \lim_{n \to \infty} SSx_n = \lim_{n \to \infty} STx_n = St \quad \text{and} \quad \lim_{n \to \infty} TSx_n = \lim_{n \to \infty} TTx_n = Tt. \]

As \( S \) and \( T \) are weakly compatible of type \((A)\), by definition
\[
\lim_{n \to \infty} G(TSx_n, SSx_n, SSx_n) \leq \lim_{n \to \infty} G(STx_n, SSx_n, SSx_n) = G(St, St, St) = 0.
\]

and
\[
\lim_{n \to \infty} G(STx_n, TTx_n, TTx_n) \leq \lim_{n \to \infty} G(TSx_n, TTx_n, TTx_n) = G(Tt, Tt, Tt) = 0.
\]

This implies,
\[ \lim_{n \to \infty} G(TSx_n, SSx_n, SSx_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} G(STx_n, TTx_n, TTx_n) = 0, \]
whenever \( \{x_n\} \) is sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). Hence, \( S \) and \( T \) are compatible of type \((A)\). ■

As a direct consequence of above Proposition, we have the following:

**Proposition 2.6.** Let \( S \) and \( T \) be \( G \)-continuous mappings from a \( G \)-metric space \((X,G)\) into itself. Then
(1) \( S \) and \( T \) are compatible of type \((A)\) iff they are weakly compatible of type \((A)\)
(2) \( S \) and \( T \) are compatible iff they are weakly compatible of type \((A)\).

**Properties of weak compatible mappings of type \((A)\) in \( G \)-metric spaces:**

**Proposition 2.7.** Let \( S \) and \( T \) be weakly compatible mappings of type \((A)\) from a \( G \)-metric space \((X,G)\) into itself. If \( Su = Tu \) for some \( u \in X \), then \( STu = SSu = TTu = TSu \).
Proof. Suppose that \( \{x_n\} \) be a sequence in \( X \) defined by \( \{x_n\} = u, n = 1, 2, 3, \) and \( Su = Tu. \) Then we have \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = Su = Tu. \) Since, \( S \) and \( T \) are weakly compatible mappings of type (A), we have
\[
G(STu, TTu, TTu) = \lim_{n \to \infty} G(STx_n, TTx_n, TTx_n) \\
\leq \lim_{n \to \infty} G(TSx_n, TTx_n, TTx_n) = G(TSu, TTu, TTu) \\
= G(TTu, TTu, TTu) = 0.
\]
Hence, we have \( STu = TTu. \) Therefore, \( STu = SSu = TTu = TSu. \)

**Proposition 2.8.** Let \( S \) and \( T \) be weakly compatible mappings of type (A) from a \( G \)-metric space \( (X, G) \) into itself. Suppose \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = u \) for some \( u \in X. \) Then we have the following:

1. \( \lim_{n \to \infty} TSx_n = Su \) if \( S \) is \( G \)-continuous at \( u \in X, \)
2. \( \lim_{n \to \infty} STx_n = Tu \) if \( T \) is \( G \)-continuous at \( u \in X, \)
3. \( STu = TSu \) and \( Su = Tu \) if \( S \) and \( T \) are \( G \)-continuous at \( u \in X. \)

Proof. (1) Suppose \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = u \) for some \( u \in X. \) Since, \( S \) is \( G \)-continuous, we have \( \lim_{n \to \infty} SSx_n = \lim_{n \to \infty} STx_n = Su. \) By (G5),
\[
G(TSx_n, Su, Su) \leq G(TSx_n, SSx_n, SSx_n) + G(SSx_n, Su, Su).
\]
As \( S \) and \( T \) are weakly compatible mappings of type (A), as \( n \to \infty \)
\[
\lim_{n \to \infty} G(TSx_n, Su, Su) \leq \lim_{n \to \infty} G(STx_n, SSx_n, SSx_n) \\
+ \lim_{n \to \infty} G(SSx_n, Su, Su) \\
= G(Su, Su, Su) + G(Su, Su, Su) = 0 + 0 = 0.
\]
This implies \( \lim_{n \to \infty} TSx_n = Su. \)

(2) The proof is similar to proof (1).

(3) Since \( T \) is \( G \)-continuous at \( u, \) we have \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = u, \) this implies \( \lim_{n \to \infty} TSx_n = \lim_{n \to \infty} TTx_n = Tu. \) By (1), since \( S \) is \( G \)-continuous at \( u, \) we also have \( \lim_{n \to \infty} TSx_n = Su. \) Hence, by uniqueness of the limit, we have \( Su = Tu \) and so by Proposition 2.7, \( STu = TSu. \)

**References**


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Received August 5, 2010; revised version December 27, 2010.