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FUZZY IDEALS OF PSEUDO-BCK ALGEBRAS

Abstract. Characterizations of fuzzy ideals of a pseudo-BCK algebra are established. Conditions for a fuzzy set to be a fuzzy ideal are given. Given a fuzzy set $\mu$, the least fuzzy ideal containing $\mu$ is constructed. The homomorphic properties of fuzzy ideals of a pseudo-BCK algebra are provided. Finally, characterizations of Noetherian pseudo-BCK algebras and Artinian pseudo-BCK algebras in terms of fuzzy ideals are given.

1. Introduction


Fuzzy ideals of BCK algebras were introduced in [16] and later were studied in [14]. See also [11] and [17]. In this paper, we consider the fuzzy ideal theory in pseudo-BCK algebras. In Section 3 we give characterizations of fuzzy ideals of a pseudo-BCK algebra. We provide conditions for a fuzzy set to be a fuzzy ideal. Given a fuzzy set $\mu$, we make the least fuzzy ideal containing $\mu$. This leads us to show that the set of fuzzy ideals of a pseudo-BCK algebra is a complete lattice. The homomorphic properties of fuzzy ideals of pseudo-BCK algebras.

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a pseudo-BCK algebra are provided. Finally, characterizations of Noetherian pseudo-BCK algebras and Artinian pseudo-BCK algebras in terms of fuzzy ideals are given in Section 4. For the convenience of the reader, in Section 2 we give the necessary material needed in the sequel, thus making our exposition self-contained.

2. Preliminaries

The notion of pseudo-BCK algebras is defined by Georgescu and Iorgulescu in [4] as follows:

**Definition 2.1.** A pseudo-BCK algebra is a structure \( A = (A; \leq, *, \circ, 0) \), where “\( \leq \)” is a binary relation on a set \( A \), “\( * \)” and “\( \circ \)” are binary operations on \( A \) and “\( 0 \)” is an element of \( A \), verifying the axioms: for all \( x, y, z \in A \),

\[
\text{(pBCK-1)} \quad (x * y) \circ (x * z) \leq z * y, \quad (x \circ y) * (x \circ z) \leq z \circ y,
\]

\[
\text{(pBCK-2)} \quad x * (x \circ y) \leq y, \quad x \circ (x * y) \leq y,
\]

\[
\text{(pBCK-3)} \quad x \leq x,
\]

\[
\text{(pBCK-4)} \quad 0 \leq x,
\]

\[
\text{(pBCK-5)} \quad (x \leq y \text{ and } y \leq x) \implies x = y,
\]

\[
\text{(pBCK-6)} \quad x \leq y \iff x * y = 0 \iff x \circ y = 0.
\]

Note that every pseudo-BCK algebra satisfying \( x * y = x \circ y \) for all \( x, y \in A \) is a BCK algebra.

The relation “\( \leq \)” is a partial order on \( A \) (see [4]), that is, \((A; \leq)\) is a poset. If \((A; \leq)\) is a chain, then \( A \) is called a pseudo-BCK chain.

**Example 2.2.** [7] Let \( A = \{0, a, b, 1\} \), where \( 0 < a < b < 1 \). We define binary operations “\( * \)” and “\( \circ \)” on \( A \) by the following tables:

\[
\begin{array}{cccc}
* & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & 0 \\
1 & 1 & b & b & 0 \\
\end{array}
\quad \quad
\begin{array}{cccc}
\circ & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & 0 \\
1 & 1 & 1 & 1 & a \\
\end{array}
\]

Then \( A = (A; \leq, *, \circ, 0) \) is a pseudo-BCK chain.

**Example 2.3.** [13] Let \( B = [0, \infty) \) and let \( \leq \) be the usual order on \( B \). Define binary operations “\( * \)” and “\( \circ \)” on \( B \) by
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$x \ast y = \begin{cases} 0 & \text{if } x \leq y, \\ \frac{2x}{\pi} \arctan(\ln(\frac{x}{y})) & \text{if } 0 < y < x, \\ x & \text{if } y = 0, \end{cases}$

$x \circ y = \begin{cases} 0 & \text{if } x \leq y, \\ xe^{-\tan(\frac{\pi y}{2})} & \text{if } y < x, \end{cases}$

for all $x, y \in B$. Then $B = (B; \leq, \ast, \circ, 0)$ is a pseudo-BCK chain.

**Lemma 2.4.** [4] Let $A$ be a pseudo-BCK algebra. Then for all $x, y, z \in A$:

(a) $(x \ast y) \circ z = (x \circ z) \ast y$,
(b) if $x \leq y$, then $x \ast z \leq y \ast z$ and $x \circ z \leq y \circ z$.

Applying (pBCK-6) and Lemma 2.4(a) it is easy to see that in a pseudo-BCK algebra the following property holds:

(1) $(\cdots (x \ast a_1) \ast \cdots) \ast a_n = 0 \iff (\cdots (x \circ a_n) \circ \cdots) \circ a_1 = 0$,

where $n$ is a natural number.

A subset $I$ of $A$ is called an ideal of a pseudo-BCK algebra $A$ if it satisfies

(I1) $0 \in I$,
(I2) if $x \ast y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\text{Id}(A)$ the set of all ideals of $A$. Obviously, $\{0\}, A \in \text{Id}(A)$. An ideal $I$ of $A$ is called proper if $I \neq A$.

**Example 2.5.** Let $A$ be the pseudo-BCK algebra from Example 2.2. Then it is easy to see that $\{0\}, \{0, a\}$ and $A$ are the only ideals of $A$.

**Proposition 2.6.** [7] Let $I$ be an ideal of a pseudo-BCK algebra $A$. Then for any $x, y \in A$, if $y \in I$ and $x \leq y$, then $x \in I$.

**Proposition 2.7.** [7] Let $A$ be a pseudo-BCK algebra and let $I \subseteq A$. Then $I$ is an ideal of $A$ if and only if it satisfies conditions (I1) and (I2') for all $x, y \in A$, if $x \circ y \in I$ and $y \in I$, then $x \in I$.

**Remark 2.8.** It is easy to prove that the intersection of an arbitrary number of ideals of a pseudo-BCK algebra $A$ is an ideal of $A$. It is also not hard to show that the union of an ascending sequence of ideals of $A$ is an ideal of $A$.

For every subset $X \subseteq A$, we denote by $(X]$ the ideal of $A$ generated by $X$, that is, $(X]$ is the smallest ideal containing $X$. By Lemma 2.2 of [7], $(\emptyset) = \{0\}$ and for every $\emptyset \neq X \subseteq A$,
(X) = \{x \in A : (\cdots (x \ast a_1) \cdots) \ast a_n = 0 \text{ for some } a_1, \ldots, a_n \in X\}
= \{x \in A : (\cdots (x \circ a_1) \circ \cdots) \circ a_n = 0 \text{ for some } a_1, \ldots, a_n \in X\}.

The set \(\text{Id}(A)\) is ordered by set-inclusion. For \(I, J \in \text{Id}(A)\) we have \(I \land J = I \cap J\) and \(I \lor J = (I \cup J)\).

**Theorem 2.9.** Let \(A\) be a pseudo-BCK algebra. Then \((\text{Id}(A); \land, \lor)\) is a complete lattice.

**Lemma 2.10.** Let \(A\) be a pseudo-BCK algebra. Let \(n\) and \(m\) be natural numbers. If

\[(x \ast a) \ast b = 0,\]
\[\cdots (a \ast a_1) \ast \cdots \ast a_n = 0,\]
\[\cdots (b \ast b_1) \ast \cdots \ast b_m = 0\]

in \(A\), then
\[\cdots (((\cdots (x \ast a_1) \ast \cdots) \ast b_1) \ast \cdots) \ast b_m = 0.\]

**Proof.** Since \((x \ast a) \ast b = 0\), we obtain by (pBCK-6) that \((x \ast a) \circ b = 0\).

Then, from Lemma 2.4(a) we have \((x \circ b) \ast a = 0\), that is, \(x \circ b \leq a\). Hence, by Lemma 2.4(b), \((x \circ b) \ast a_1 \leq a \ast a_1\). Next, again by Lemma 2.4(b), we have \(((x \circ b) \ast a_1) \ast a_2 \leq (a \ast a_1) \ast a_2\). Repeating the process we get
\[\cdots (((\cdots (x \circ b) \ast a_1) \ast \cdots) \ast a_n \leq (\cdots (a \ast a_1) \ast \cdots) \ast a_n = 0.\]

Applying (pBCK-4) and (pBCK-5) we obtain \(\cdots (((\cdots (x \circ b) \ast a_1) \cdots) \ast a_n = 0\).

From Lemma 2.4(a) we deduce that \(((\cdots (x \ast a_1) \ast \cdots) \ast a_n) \circ b = 0\), that is, \((\cdots (x \ast a_1) \ast \cdots) \ast a_n \leq b\). Hence, by Lemma 2.4(b),
\[\cdots (((\cdots (x \ast a_1) \ast \cdots) \ast a_n) \ast b_1) \ast \cdots) \ast b_m \leq (\cdots (b \ast b_1) \ast \cdots) \ast b_m = 0.\]

Therefore \(\cdots (((\cdots (x \ast a_1) \ast \cdots) \ast a_n) \ast b_1) \ast \cdots) \ast b_m = 0.\]

3. General fuzzy concepts. Fuzzy ideals

We now review some fuzzy concepts. First, for \(\Gamma \subseteq [0, 1]\) we define \(\land \Gamma = \inf \Gamma\) and \(\lor \Gamma = \sup \Gamma\). Obviously, if \(\Gamma = \{\alpha, \beta\}\), then \(\alpha \land \beta = \min \{\alpha, \beta\}\) and \(\alpha \lor \beta = \max \{\alpha, \beta\}\). The algebra \(([0, 1]; \land, \lor)\) is a complete lattice. Recall that a fuzzy set of \(A\) is a function \(\mu : A \to [0, 1]\).

For any fuzzy sets \(\mu\) and \(\nu\) of \(A\), we define
\[
\mu \leq \nu \text{ iff } \mu(x) \leq \nu(x) \text{ for all } x \in A.
\]

It is easy to check that this relation is an order relation in the set of fuzzy sets of \(A\).

**Definition 3.1.** Let \(A\) and \(B\) be any two sets, \(\mu\) be any fuzzy set of \(A\) and \(f : A \to B\) be any function. Set \(f^{-1}(y) = \{x \in A : f(x) = y\}\) for \(y \in B\).
The fuzzy set $\nu$ of $B$ defined by

$$\nu(y) = \begin{cases} \bigvee\{\mu(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in B$, is called the image of $\mu$ under $f$ and is denoted by $f(\mu)$.

**Definition 3.2.** Let $A$ and $B$ be any two sets, $f : A \to B$ be any function and $\nu$ be any fuzzy set of $f(A)$. The fuzzy set $\mu$ of $A$ defined by

$$\mu(x) = \nu(f(x))$$

for all $x \in A$ is called the preimage of $\nu$ under $f$ and is denoted by $f^{-1}(\nu)$.

We say that $\mu$ is a fuzzy set of a pseudo-BCK algebra $A$ if $\mu$ is a fuzzy set of $A$.

**Definition 3.3.** A fuzzy set $\mu$ of a pseudo-BCK algebra $A$ is called a fuzzy ideal of $A$ if it satisfies for all $x, y \in A$:

(d1) $\mu(0) \geq \mu(x)$,
(d2) $\mu(x) \geq \mu(x * y) \wedge \mu(y)$.

**Proposition 3.4.** Let $\mu$ be a fuzzy ideal of a pseudo-BCK algebra $A$. Then, for any $x, y \in A$, if $x \leq y$, then $\mu(x) \geq \mu(y)$.

**Proof.** If $x \leq y$, then $x * y = 0$. Hence, by (d2), we have $\mu(x) \geq \mu(x * y) \wedge \mu(y) = \mu(0) \wedge \mu(y) = \mu(y)$.

Denote by $\mathcal{FI}(A)$ the set of fuzzy ideals of a pseudo-BCK algebra $A$.

**Example 3.5.** Let $A$ be the pseudo-BCK algebra from Example 2.2. Let $0 \leq \alpha_3 < \alpha_2 < \alpha_1 \leq 1$. Define a fuzzy set $\mu$ of $A$ by

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x = 0, \\ \alpha_2 & \text{if } x = a, \\ \alpha_3 & \text{if } x \in \{b, 1\}. \end{cases}$$

It is easily checked that $\mu$ satisfies (d1) and (d2). Thus $\mu \in \mathcal{FI}(A)$.

**Example 3.6.** Let $I$ be an ideal of a pseudo-BCK algebra $A$ and let $\alpha, \beta \in [0, 1]$, with $\alpha > \beta$. Define $\mu_I^{\alpha,\beta}$ as follows:

$$\mu_I^{\alpha,\beta}(x) := \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise}. \end{cases}$$

We denote $\mu_I^{\alpha,\beta} = \mu$. Since $0 \in I$, $\mu(0) = \alpha \geq \mu(x)$ for all $x \in A$. To prove (d2), let $x, y \in A$. If $x \in I$, then $\mu(x) = \alpha \geq \mu(x * y) \wedge \mu(y)$. Suppose now that $x \notin I$. By the definition of ideal, $x * y \notin I$ or $y \notin I$. Therefore, $\mu(x * y) \wedge \mu(y) = \beta = \mu(x)$. Thus $\mu$ is a fuzzy ideal of $A$. 

In particular the characteristic function $\chi_{I}$ of $I$:

$$\chi_{I}(x) = \begin{cases} 
1 & \text{if } x \in I, \\
0 & \text{otherwise}
\end{cases}$$

is a fuzzy ideal of $A$.

**Proposition 3.7.** A fuzzy set $\mu$ of a pseudo-BCK algebra $A$ is a fuzzy ideal of $A$ if and only if it satisfies (d1) and

$$(d2') \; \mu(x) \geq \mu(x \circ y) \land \mu(y) \text{ for all } x, y \in A.$$  

**Proof.** It suffices to prove that if (d2) is satisfied, then (d2') is also satisfied. The proof of the converse of this implication is analogous. From (pBCK-2) we know that $x \mathbin{\ast} (x \circ y) \leq y$. Thus, by Proposition 3.4, $\mu(y) \leq \mu((x \mathbin{\ast} (x \circ y)))$. Hence

$$\mu(x \circ y) \land \mu(y) \leq \mu(x \circ y) \land \mu(x \circ (x \circ y)).$$

By (d2),

$$\mu(x \circ y) \land \mu(x \circ (x \circ y)) \leq \mu(x).$$

By (2) and (3) we obtain $\mu(x \circ y) \land \mu(y) \leq \mu(x)$, that is, (d2') holds.

**Proposition 3.8.** A fuzzy set $\mu$ of a pseudo-BCK algebra $A$ is a fuzzy ideal of $A$ if and only if it satisfies (d1) and

$$(d3) \; \text{for all } x, y, z \in A, \text{ if } (x \mathbin{\ast} y) \mathbin{\ast} z = 0, \text{ then } \mu(x) \geq \mu(y) \land \mu(z).$$

**Proof.** Let $\mu \in \mathcal{FI}(A)$ and let $x, y, z \in A$. Suppose that $(x \mathbin{\ast} y) \mathbin{\ast} z = 0$. Since $\mu$ is a fuzzy ideal, we have $\mu(x \mathbin{\ast} y) \geq \mu((x \mathbin{\ast} y) \mathbin{\ast} z) \land \mu(z) = \mu(0) \land \mu(z) = \mu(z)$ and $\mu(x) \geq \mu(x \mathbin{\ast} y) \land \mu(y)$. Therefore, $\mu(x) \geq \mu(y) \land \mu(z)$.

Conversely, let $\mu$ satisfy (d3). Applying (pBCK-3) we have $(x \mathbin{\ast} y) \mathbin{\ast} z = 0$, where $z = x \mathbin{\ast} y$. By (d3), $\mu(x) \geq \mu(y) \land \mu(z) = \mu(y) \land \mu(x \mathbin{\ast} y)$. Then $\mu$ satisfies (d2) and hence $\mu \in \mathcal{FI}(A)$.

It is easy to prove by induction the following.

**Proposition 3.9.** Let $\mu$ be a fuzzy set satisfying (d1) of a pseudo-BCK algebra $A$. Then $\mu$ is a fuzzy ideal if and only if for any $a_1, \ldots, a_n \in A$ ($n \geq 2$),

$$(\cdots (x \mathbin{\ast} a_1) \mathbin{\ast} \cdots) \mathbin{\ast} a_n = 0 \text{ implies } \mu(x) \geq \mu(a_1) \land \ldots \land \mu(a_n).$$

**Theorem 3.10.** Let $\mu$ be a fuzzy set of a pseudo-BCK algebra $A$. Then $\mu \in \mathcal{FI}(A)$ if and only if its nonempty level subset

$$U(\mu; \alpha) := \{x \in A : \mu(x) \geq \alpha\}$$

is an ideal of $A$ for all $\alpha \in [0, 1]$. 
Proof. Assume that \( \mu \in FI(\mathcal{A}) \) and let \( \alpha \in [0, 1] \) be such that \( U(\mu; \alpha) \neq \emptyset \). Then \( \mu(x_0) \geq \alpha \) for some \( x_0 \in \mathcal{A} \). Since \( \mu(0) \geq \mu(x_0) \), we have \( 0 \in U(\mu; \alpha) \).

Let \( x, y \in \mathcal{A} \) be such that \( x \ast y, y \in U(\mu; \alpha) \). Then \( \mu(x \ast y) \geq \alpha \) and \( \mu(y) \geq \alpha \).

It follows from (d2) that
\[
\mu(x) \geq \mu(x \ast y) \land \mu(y) \geq \alpha
\]
so that \( x \in U(\mu; \alpha) \). Therefore \( U(\mu; \alpha) \) is an ideal of \( \mathcal{A} \).

Conversely, suppose that for each \( \alpha \in [0, 1] \), \( U(\mu; \alpha) = \emptyset \) or \( U(\mu; \alpha) \) is an ideal of \( \mathcal{A} \). If (d1) is not valid, then there exists \( x_0 \in \mathcal{A} \) such that \( \mu(0) < \mu(x_0) := \beta \). Then \( U(\mu; \beta) \neq \emptyset \) and by assumption, \( U(\mu; \beta) \) is an ideal of \( \mathcal{A} \). Hence \( 0 \in U(\mu; \beta) \) and consequently, \( \mu(0) \geq \beta \). This is a contradiction and (d1) is valid. Now assume that (d2) does not hold. Then there are \( a, b \in \mathcal{A} \) such that \( \mu(a) < \mu(a \ast b) \land \mu(b) \). Taking
\[
\beta = \frac{1}{2}(\mu(a) + \mu(a \ast b) \land \mu(b)),
\]
we get \( \mu(a) < \beta < \mu(a \ast b) \land \mu(b) \leq \mu(a \ast b) \) and \( \beta < \mu(b) \). Therefore \( a \ast b, b \in U(\mu; \beta) \) but \( a \notin U(\mu; \beta) \). This is impossible, and \( \mu \) is a fuzzy ideal of \( \mathcal{A} \). \( \blacksquare \)

Example 3.11. Let \( \mu \) be as in Example 3.5. One can easily check that for all \( \alpha \in [0, 1] \) we have:
\[
U(\mu; \alpha) = \begin{cases} 
\emptyset & \text{if } \alpha > \alpha_1, \\
\{0\} & \text{if } \alpha_2 < \alpha \leq \alpha_1, \\
\{0, a\} & \text{if } \alpha_3 < \alpha \leq \alpha_2, \\
A & \text{if } \alpha \leq \alpha_3.
\end{cases}
\]

Since \( \{0\}, \{0, a\} \) and \( A \) are all ideals of \( \mathcal{A} \), this is an another proof (by Theorem 3.10) that \( \mu \) is a fuzzy ideal of \( \mathcal{A} \).

Corollary 3.12. If \( \mu \) is a fuzzy ideal of a pseudo-BCK algebra \( \mathcal{A} \), then the set
\[
A_b := \{x \in \mathcal{A} : \mu(x) \geq \mu(b)\}
\]
is an ideal of \( \mathcal{A} \) for every \( b \in \mathcal{A} \).

By Corollary 3.12, we have the following.

Corollary 3.13. If \( \mu \) is a fuzzy ideal of a pseudo-BCK algebra \( \mathcal{A} \), then the set
\[
A_\mu := \{x \in \mathcal{A} : \mu(x) = \mu(0)\}
\]
is an ideal of \( \mathcal{A} \).

The following example shows that the converse of Corollary 3.13 does not hold.
**Example 3.14.** Let $A$ be a pseudo-BCK algebra. Define a fuzzy set $\mu$ of $A$ by

$$
\mu(x) = \begin{cases} 
0.4 & \text{if } x = 0, \\
0.6 & \text{if } x \neq 0.
\end{cases}
$$

Then $A_\mu = \{0\}$ and it is an ideal of $A$ but $\mu \notin \mathcal{FI}(A)$, because $\mu$ does not satisfy (d1).

**Lemma 3.15.** Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be a strictly ascending sequence of ideals of a pseudo-BCK algebra $A$ and $(t_n)$ be a strictly decreasing sequence in $(0,1)$. Let $\mu$ be the fuzzy set of $A$ defined by

$$
\mu(x) = \begin{cases} 
0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\
t_n & \text{if } x \in I_n - I_{n-1} \text{ for } n = 1, 2, \ldots,
\end{cases}
$$

where $I_0 = \emptyset$. Then $\mu$ is a fuzzy ideal of $A$.

**Proof.** Let $I = \bigcup_{n \in \mathbb{N}} I_n$. By Remark 2.8, $I$ is an ideal of $A$. Obviously, $\mu(0) = t_1 \geq \mu(x)$ for all $x \in A$, that is, (d1) holds. Now we show that $\mu$ satisfies (d2). Let $x, y \in A$. We have two cases.

Case 1: $x \notin I$.

Then $x \ast y \notin I$ or $y \notin I$. Therefore $\mu(x \ast y) \wedge \mu(y) = 0 = \mu(x)$.

Case 2: $x \in I_n - I_{n-1}$ for some $n = 1, 2, \ldots$.

Then $x \ast y \notin I_{n-1}$ or $y \notin I_{n-1}$. Hence $\mu(x \ast y) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(x \ast y) \wedge \mu(y) \leq t_n = \mu(x)$.

Thus (d2) is also satisfied and consequently, $\mu$ is a fuzzy ideal of $A$. ■

Let $\mu_t \in \mathcal{FI}(A)$ for $t \in T$. The meet $\bigwedge_{t \in T} \mu_t$ is defined as follows:

$$
\left( \bigwedge_{t \in T} \mu_t \right)(x) = \bigwedge \{ \mu_t(x) : t \in T \}.
$$

**Theorem 3.16.** Let $\mu_t \in \mathcal{FI}(A)$ for $t \in T$. Then $\bigwedge_{t \in T} \mu_t \in \mathcal{FI}(A)$.

**Proof.** Let $\mu = \bigwedge_{t \in T} \mu_t$. Then, by (d1),

$$
\mu(0) = \bigwedge \{ \mu_t(0) : t \in T \} \geq \bigwedge \{ \mu_t(x) : t \in T \} = \mu(x)
$$

for all $x \in A$. Let $x, y \in A$. Since $\mu_t \in \mathcal{FI}(A)$, we have $\mu_t(x) \geq \mu_t(x \ast y) \wedge \mu_t(y)$. Hence

$$
\bigwedge \{ \mu_t(x) : t \in T \} \geq \bigwedge \{ \mu_t(x \ast y) \wedge \mu_t(y) : t \in T \}
$$

$$
= \bigwedge \{ \mu_t(x \ast y) : t \in T \} \wedge \bigwedge \{ \mu_t(y) : t \in T \}.
$$

Consequently, $\mu(x) \geq \mu(x \ast y) \wedge \mu(y)$ and therefore $\mu \in \mathcal{FI}(A)$. ■

**Remark 3.17.** Since $U(\bigwedge_{t \in T} \mu_t; \alpha) = \bigcap_{t \in T} U(\mu_t : \alpha)$, we see that Theorem 3.16 follows from Remark 2.8 and Theorem 3.10.
Let $f$ be a fuzzy set of $A$. A fuzzy ideal $\mu$ of $A$ is said to be generated by $f$ if $f \leq \mu$ and for any fuzzy ideal $\nu$ of $A$, $f \leq \nu$ implies $\mu \leq \nu$. The fuzzy ideal generated by $f$ will be denoted by $[f]$. The fuzzy ideal $[f]$ can be defined equivalently as follows:

$$(f) = \bigwedge\{\nu \in F\mathcal{I}(A) : \nu \geq f\}.$$  

We have a simple theorem.

**Theorem 3.18.** Let $f$ and $g$ be fuzzy sets of $A$. The following properties hold:

(a) $f \leq g$ implies $[f] \leq [g]$,

(b) if $f \in F\mathcal{I}(A)$, then $[f] = f$.

**Theorem 3.19.** Let $f$ be a fuzzy set of a pseudo-BCK algebra $A$ and let $\mu$ be a fuzzy set of $A$ defined for all $x \in A$ by

$$\mu(x) = \bigvee \{f(a_1) \land \cdots \land f(a_n) : (\cdots (x \ast a_1) \ast \cdots) \ast a_n = 0$$

and $a_1, \ldots, a_n \in A\}.$

Then $\mu = [f]$.

**Proof.** It is easy to see that $\mu(0) \geq \mu(x)$ for all $x \in A$. Now we prove that $\mu$ satisfies (d3). Suppose that $(x \ast a) \ast b = 0$, where $x, a, b \in A$. Let $k \in \mathbb{N}$. By the definition of $\mu$, we can select $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ such that

$$(\cdots (a \ast a_1) \ast \cdots) \ast a_n = 0,$$

$$(\cdots (b \ast b_1) \ast \cdots) \ast b_m = 0,$$

$$f(a_1) \land \cdots \land f(a_n) > \mu(a) - \frac{1}{k},$$

$$f(b_1) \land \cdots \land f(b_m) > \mu(b) - \frac{1}{k}.$$  

From Lemma 2.10 it follows that $(\cdots ((\cdots (x \ast a_1) \ast \cdots) \ast a_n) \ast b_1) \ast \cdots) \ast b_m = 0$. Then

$$\mu(x) \geq f(a_1) \land \cdots \land f(a_n) \land f(b_1) \land \cdots \land f(b_m) > \left(\mu(a) - \frac{1}{k}\right) \land \left(\mu(b) - \frac{1}{k}\right).$$  

Hence $\mu(x) \geq \mu(a) \land \mu(b)$ and by Proposition 3.8, $\mu \in F\mathcal{I}(A)$.

Applying (pBCK-3) we get $x \ast x = 0$. From this we see that $f(x) \leq \mu(x)$. Thus $f \leq \mu$. Finally, suppose $\nu$ is a fuzzy ideal of $A$ such that $f \leq \nu$. Then for any $x \in A$ we obtain

$$\mu(x) = \bigvee \{f(a_1) \land \cdots \land f(a_n) : (\cdots (x \ast a_1) \ast \cdots) \ast a_n = 0$$

and $a_1, \ldots, a_n \in A\}$$

$$\leq \bigvee \{\nu(a_1) \land \cdots \land \nu(a_n) : (\cdots (x \ast a_1) \ast \cdots) \ast a_n = 0 \text{ and } a_1, \ldots, a_n \in A\}.$$  

and by Proposition 3.9,
\[ \bigvee \{ \nu(a_1) \land \cdots \land \nu(a_n) : (\cdots (x*a_1) \cdots )^n a_n = 0 \text{ and } a_1, \ldots, a_n \in A \} \leq \nu(x). \]
Therefore \( \mu(x) \leq \nu(x) \) for all \( x \in A \). Consequently, \( \mu \leq \nu \). Thus \( \mu \) is the fuzzy ideal generated by \( f \), that is, \( \mu = [f] \).

**Remark 3.20.** Let \( f \) be a fuzzy set of a pseudo-BCK algebra \( A \). From Theorem 3.19 and (1) we have
\[
(f)(x) = \bigvee \{ f(a_1) \land \cdots \land f(a_n) : (\cdots (x \land a_1) \cdots )^n a_n = 0 \text{ and } a_1, \ldots, a_n \in A \}
\]
\[ = \bigvee \{ f(a_1) \land \cdots \land f(a_n) : (\cdots (x \lor a_1) \cdots )^n a_n = 0 \text{ and } a_1, \ldots, a_n \in A \}
\]
for all \( x \in A \).

**Example 3.21.** Let \( A \) be the pseudo-BCK algebra from Example 2.2. Define a fuzzy set \( f \) of \( A \) by
\[
f(x) = \begin{cases} 
0.7 & \text{if } x = 0, \\
0.3 & \text{if } x \in \{a, b\}, \\
0 & \text{if } x = 1.
\end{cases}
\]
Then the fuzzy ideal \( \mu = [f] \) generated by \( f \) is as follows:
\[
\mu(x) = \begin{cases} 
0.7 & \text{if } x = 0, \\
0.3 & \text{if } x \in \{a, b, 1\}.
\end{cases}
\]

For \( \mu, \nu \in \mathcal{FI}(A) \) let \( \mu \lor \nu \) denote the join of \( \mu \) and \( \nu \), that is, \( \mu \lor \nu = [f] \), where \( f \) is the fuzzy set of \( A \) defined by \( f(x) = \mu(x) \lor \nu(x) \) for all \( x \in A \).

From Theorem 3.16 or from Theorems 2.9 and 3.10 we obtain

**Theorem 3.22.** Let \( A \) be a pseudo-BCK algebra. Then \( (\mathcal{FI}(A) ; \land, \lor) \) is a complete lattice.

The following two theorems give the homomorphic properties of fuzzy ideals.

**Theorem 3.23.** Let \( A \) and \( B \) be pseudo-BCK algebras and let \( f : A \to B \) be a surjective homomorphism and \( \nu \in \mathcal{FI}(B) \). Then \( f^{-1}(\nu) \in \mathcal{FI}(A) \).

**Proof.** Let \( x \in A \). Since \( f(x) \in B \) and \( \nu \in \mathcal{FI}(B) \), we have \( \nu(0) \geq \nu(f(x)) = (f^{-1}(\nu))(x) \), but \( \nu(0) = \nu(f(0)) = (f^{-1}(\nu))(0) \). Thus we get \( (f^{-1}(\nu))(0) \geq (f^{-1}(\nu))(x) \) for any \( x \in A \), that is, \( f^{-1}(\nu) \) satisfies (d1).

Now let \( x, y \in A \). Since \( \nu \in \mathcal{FI}(B) \), we have
\[
\nu(f(x)) \geq \nu(f(x) \ast f(y)) \land \nu(f(y)) = \nu(f(x \ast y)) \land \nu(f(y))
\]
and hence $f^{-1}(\nu)(x) \geq f^{-1}(\nu)(x * y) \land f^{-1}(\nu)(y)$. Consequently, $f^{-1}(\nu) \in \mathcal{FI}(A)$. ■

**Lemma 3.24.** Let $A$ and $B$ be pseudo-BCK algebras and let $f : A \to B$ be a homomorphism and $\mu \in \mathcal{FI}(A)$. Then, if $\mu$ is constant on $\ker f = f^{-1}(0)$, then $f^{-1}(\mu(f)) = \mu$.

**Proof.** Let $x \in A$ and $f(x) = y$. Hence

$$(f^{-1}(\mu(f))) (x) = (\mu(f))(f(x)) = (\mu(f))(y) = \bigvee \{ \mu(a) : a \in f^{-1}(y) \}.$$  

For all $a \in f^{-1}(y)$, we have $f(a) = f(x)$. Then by (pBCK-3), $f(a) * f(x) = 0$. Hence $f(a*x) = 0$, that is, $a*x \in \ker f$. Thus $\mu(a*x) = \mu(0)$. Therefore, $\mu(a) \geq \mu(a*x) \land \mu(x) = \mu(0) \land \mu(x) = \mu(x)$. Similarly, $\mu(x) \geq \mu(a)$. Hence $\mu(x) = \mu(a)$. Thus

$$(f^{-1}(\mu(f))) (x) = \bigvee \{ \mu(a) : a \in f^{-1}(y) \} = \mu(x),$$

i.e., $f^{-1}(\mu(f)) = \mu$. ■

**Theorem 3.25.** Let $A$ and $B$ be pseudo-BCK algebras and let $f : A \to B$ be a surjective homomorphism and $\mu \in \mathcal{FI}(A)$ be such that $A_{\mu} \supseteq \ker f$. Then $\mu(f) \in \mathcal{FI}(B)$.

**Proof.** Since $\mu$ is a fuzzy ideal of $A$ and $0 \in f^{-1}(0)$, we have

$$(\mu(f))(0) = \bigvee \{ \mu(a) : a \in f^{-1}(0) \} = \mu(0) \geq \mu(x)$$

for any $x \in A$. Hence

$$(\mu(f))(0) \geq \bigvee \{ \mu(x) : x \in f^{-1}(y) \} = (\mu(f))(y)$$

for any $y \in B$. Thus $\mu(f)$ satisfies (d1). Suppose that

$$f(\mu)(x_B) < f(\mu)(x_B * y_B) \land f(\mu)(y_B)$$

for some $x_B, y_B \in B$. Since $f$ is surjective, there are $x_A, y_A \in A$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Hence

$$f(\mu)(f(x_A)) < f(\mu)(f(x_A * y_A)) \land f(\mu)(f(y_A)).$$

Therefore

$$f^{-1}(\mu(f))(x_A) < f^{-1}(\mu(f))(x_A * y_A) \land f^{-1}(\mu(f))(y_A)).$$

Since $A_\mu \supseteq \ker f$, $\mu$ is constant on $\ker f$. Hence, by Lemma 3.24, we get

$$\mu(x_A) < \mu(x_A * y_A) \land \mu(y_A),$$

which is a contradiction with the fact that $\mu$ is a fuzzy ideal. Thus $\mu(f) \in \mathcal{FI}(B)$. ■
4. Fuzzy characterizations of Noetherian and Artinian pseudo-BCK algebras

In this section we characterize Noetherian pseudo-BCK algebras and Artinian pseudo-BCK algebras using some fuzzy concepts, in particular, fuzzy ideals. In the beginning we give some definitions.

A pseudo-BCK algebra $A$ is called Noetherian if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of $A$ there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$. A pseudo-BCK algebra $A$ is called Artinian if for every descending sequence $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of $A$ there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$.

Obviously, every finite pseudo-BCK algebra is Noetherian and Artinian.

**Example 4.1.** Let $B$ be the pseudo-BCK algebra from Example 2.3. Observe that $\text{Id}(B) = \{\{0\}, B\}$. Let $I \neq \{0\}$ be an ideal of $B$. Then there is an element $x \in I - \{0\}$. Since $(2x) \circ x = 2x < x \in I$, we conclude that $(2x) \circ x \in I$ and hence $2x \in I$. From this it follows that $2^n x \in I$ for each $n \in \mathbb{N}$. Let $y \in B$. Clearly, $\frac{y}{x} < 2^m$ for some $m \in \mathbb{N}$. Therefore $y < 2^m x \in I$. Then $y \in I$ and consequently, $I = B$. Thus $\text{Id}(B) = \{\{0\}, B\}$ and hence $B$ is both Noetherian and Artinian.

**Example 4.2.** Let $(P; \leq)$ be a poset with a least element 0. For $x, y \in P$, we put

$$x \ast y = x \circ y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise.} \end{cases}$$

Then $(P; \leq, \ast, \circ, 0)$ is a pseudo-BCK algebra. It is easy to see that for every $x \in P$ the subset $[0, x] = \{y \in P : y \leq x\}$ is an ideal of $P$.

For $P = \mathbb{Q}$ (the rational numbers) we have

(4) \[ [0, 1] \subset [0, 2] \subset \cdots \subset [0, n] \subset \cdots \]

and

(5) \[ [0, 1] \supset \left[0, \frac{1}{2}\right] \supset \cdots \supset \left[0, \frac{1}{n}\right] \supset \cdots . \]

Hence the pseudo-BCK algebra $(\mathbb{Q}; \leq, \ast, \circ, 0)$ is not Noetherian and also not Artinian.

Now we consider $P = \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. From (4) it follows that $\mathcal{N} = (\mathbb{N}_0; \leq, \ast, \circ, 0)$ is not Noetherian. Since every proper ideal of $\mathcal{N}$ is a set of the form $\{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$, we conclude that $\mathcal{N}$ is an Artinian pseudo-BCK algebra.

**Theorem 4.3.** Let $\mathcal{A}$ be a pseudo-BCK algebra. The following statements are equivalent:
(a) \( \mathcal{A} \) is Noetherian,
(b) for each fuzzy ideal \( \mu \) of \( \mathcal{A} \), \( \text{Im}(\mu) = \{ \mu(x) : x \in A \} \) is a well-ordered set.

**Proof.** (a)⇒(b): Assume that \( \mathcal{A} \) is Noetherian and \( \mu \) is a fuzzy ideal of \( \mathcal{A} \) such that \( \text{Im}(\mu) \) is not a well-ordered subset of \([0,1]\). Then there exists a strictly decreasing sequence \( (\mu(x_n)) \), where \( x_n \in \mathcal{A} \). Let \( t_n = \mu(x_n) \) and \( U_n = U(\mu; t_n) = \{ x \in \mathcal{A} : \mu(x) \geq t_n \} \). Then, by Theorem 3.10, \( U_n \) is an ideal of \( \mathcal{A} \) for every \( n \in \mathbb{N} \). So \( U_1 \subset U_2 \subset \ldots \) is a strictly ascending sequence of ideals of \( \mathcal{A} \). This contradicts the assumption that \( \mathcal{A} \) is Noetherian. Therefore \( \text{Im}(\mu) \) is a well-ordered set for each fuzzy ideal \( \mu \) of \( \mathcal{A} \).

(b)⇒(a): Assume that (b) is true. Suppose that \( \mathcal{A} \) is not Noetherian. Then there exists a strictly ascending sequence \( I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots \) of fuzzy ideals of \( \mathcal{A} \). Let \( \mu \) be a fuzzy set of \( \mathcal{A} \) by
\[
\mu(x) = \begin{cases} 
0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\
\frac{1}{n} & \text{if } x \in I_n - I_{n-1} \text{ for } n = 1, 2, \ldots,
\end{cases}
\]
where \( I_0 = \emptyset \). By Lemma 3.15, \( \mu \in \mathcal{FI}(\mathcal{A}) \), but \( \text{Im}(\mu) \) is not a well-ordered set, which is a contradiction. Therefore \( \mathcal{A} \) is Noetherian and the proof is complete.

**Corollary 4.4.** Let \( \mathcal{A} \) be a pseudo-BCK algebra. If for every fuzzy ideal \( \mu \) of \( \mathcal{A} \), \( \text{Im}(\mu) \) is a finite set, then \( \mathcal{A} \) is Noetherian.

**Theorem 4.5.** Let \( \mathcal{A} \) be a pseudo-BCK algebra and let \( T = \{t_1, t_2, \ldots\} \cup \{0\} \), where \( (t_n) \) is a strictly decreasing sequence in \((0,1)\). Then the following conditions are equivalent:
(a) \( \mathcal{A} \) is Noetherian,
(b) for each fuzzy ideal \( \mu \) of \( \mathcal{A} \), if \( \text{Im}(\mu) \subseteq T \), then there exists \( k \in \mathbb{N} \) such that \( \text{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\} \).

**Proof.** (a)⇒(b): Assume that \( \mathcal{A} \) is Noetherian. Let \( \mu \) be a fuzzy ideal of \( \mathcal{A} \) such that \( \text{Im}(\mu) \subseteq T \). From Theorem 4.3 we know that \( \text{Im}(\mu) \) is a well-ordered subset of \([0,1]\). Thus there exists \( k \in \mathbb{N} \) such that \( \text{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\} \).

(b)⇒(a): Assume that (b) is true. Suppose that \( \mathcal{A} \) is not Noetherian. Then there exists a strictly ascending sequence \( I_1 \subset I_2 \subset \ldots \) of ideals of \( \mathcal{A} \). Define a fuzzy set \( \mu \) of \( A \) by
\[
\mu(x) = \begin{cases} 
0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\
t_n & \text{if } x \in I_n - I_{n-1} \text{ for } n = 1, 2, \ldots,
\end{cases}
\]
where \( I_0 = \emptyset \). By Lemma 3.15, \( \mu \) is a fuzzy ideal of \( \mathcal{A} \). This contradicts our assumption. Thus \( \mathcal{A} \) is Noetherian.
Theorem 4.6. Let $A$ be a pseudo-BCK algebra and let $T = \{t_1, t_2, \ldots\} \cup \{0, 1\}$, where $(t_n)$ is a strictly increasing sequence in $(0, 1)$. Then the following conditions are equivalent:

(a) $A$ is Artinian,
(b) for each fuzzy ideal $\mu$ of $A$, if $\text{Im}(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $\text{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0, 1\}$.

Proof. (a) $\Rightarrow$ (b). On the contrary assume that $t_{i_1} < t_{i_2} < \cdots < t_{i_m} < \cdots$ is a strictly increasing sequence of elements of $\text{Im}(\mu)$. Let $U_m = U(\mu; t_{i_m})$ for $m = 1, 2, \ldots$. It is easy to see that $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_m \supseteq \cdots$ is a strictly descending sequence of ideals of $A$. This contradicts the assumption that $A$ is Artinian.

(b) $\Rightarrow$ (a) Assume that (b) is true. Suppose that $A$ is not Artinian. Then there exists a strictly descending sequence $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of ideals of $A$. Define a fuzzy set $\mu$ of $A$ by

$$\mu(x) = \begin{cases} 
0 & \text{if } x \notin I_1, \\
t_n & \text{if } x \in I_n - I_{n+1} \text{ for } n = 1, 2, \ldots, \\
1 & \text{if } x \in \bigcap\{I_n : n \in \mathbb{N}\}.
\end{cases}$$

Obviously, $\mu(0) = 1 \geq \mu(x)$ for all $x \in A$, that is, (d1) holds. Now we show that $\mu$ satisfies (d2). Let $x, y \in A$. We have three cases.

Case 1: $x \notin I_1$. Then $x \ast y \notin I_1$ or $y \notin I_1$. Therefore $\mu(x \ast y) \land \mu(y) = 0 = \mu(x)$.

Case 2: $x \in I_n - I_{n+1}$ for some $n = 1, 2, \ldots$. Then $x \ast y \notin I_{n+1}$ or $y \notin I_{n+1}$. Hence $\mu(x \ast y) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(x \ast y) \land \mu(y) \leq t_n = \mu(x)$.

Case 3: $x \in \bigcap\{I_n : n \in \mathbb{N}\}$. Obvious.

Thus $\mu$ is a fuzzy ideal of $A$. This contradicts our assumption. Thus $A$ is Artinian. ■

Corollary 4.7. Let $A$ be a pseudo-BCK algebra. If for every fuzzy ideal $\mu$ of $A$, $\text{Im}(\mu)$ is a finite set, then $A$ is Artinian.

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References


